Michele Emmer
Marco Abate *Editors*

Imagine Math 8

Dreaming Venice

Springer

The Rise of Abstractionism: Art and Mathematics

Marco Andreatta

1 "Breaking the Ties with the Concrete and Tangible"

In the second half of the nineteenth century, many spectres were haunting Europe. Besides the spectre of communism, celebrated by the German philosophers K. Marx and F. Engels, there was the one of non-Euclidean geometry. It was first glimpsed by the Russian mathematician N.I. Lobačevskij and the Hungarian J. Bolyai who are considered the pioneers of the field, under the influential supervision of the *Princeps mathematicorum* F. Gauss. These two anarchists of the philosophy of geometry were actually preceded by earlier visionaries, the most significant was probably the Italian Jesuit priest Girolamo Saccheri, who wrote the book "Euclides ab omni naevo vindicatus" (Euclid Freed of Every Flaw (1733)) which languished in obscurity until it was rediscovered by Eugenio Beltrami.

The Manifesto of a new science of the space, which includes non-Euclidean geometry as well, is the celebrated lecture of Bernhard Riemann (1826–1866) in 1856, published posthumously in 1867, titled *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (On the Hypotheses which lie at the Bases of Geometry). Together with the *Manifesto of the Communist Party* (1848, Marx and Engel), *the Origin of Species* (1859, C. Darwin) and *the Interpretation of Dreams* (1899, S. Freud), it is one of the most influential papers in all fields of human cultural activities.

Riemann concludes the lecture as follows: *The question of the validity of the hypotheses of geometry is bound up with the question of the ground of the metric relations of space. In this last question, which we may still regard as belonging to the doctrine of space, is found the application of the remark made above; that in*

M. Andreatta (✉)
Dipartimento di Matematica, Universitá di Trento, Trento, Italy
e-mail: marco.andreatta@unitn.it

M. Emmer, M. Abate (eds.), *Imagine Math 8*,
https://doi.org/10.1007/978-3-030-92690-8_5

a discrete manifoldness, the ground of its metric relations is given in the notion of it, while in a continuous manifoldness, this ground must come from outside. Either therefore the reality which underlies space must form a discrete manifoldness, or we must seek **the ground of its metric relations outside it, in binding forces which act upon it.** ... *This leads us into the domain of another science, of physics, into which the object of this work does not allow us to go today.*

The lecture can be considered as a starting point for a more abstract study of geometry; it is full of philosophical, in a broader sense, implications. First of all, the observation, originally attributed to Leibniz, that geometry is a **science of the space**, rather than a science of objects contained in it. The objects are described by their reciprocal distance, their measures and shapes, using the *metric relations* defined on the space. Space can be of higher dimension and not necessarily Euclidian but possibly curved by the choice of a non-flat Riemann metric.

Alexander Grothendieck in *Recoltes et semailles* remarks the following: *He* [Riemann] *observed that the ultimate structure of the space is probably discrete; the continuous representation that we do could be an ultra simplification of a more complex reality. That is, for human mind continuous is easier to understand rather than discrete; as a consequence the first is used as an approximation to understand the second. This was an incredibly sharp observation for a mathematician, in a period in which the Euclidian model of the space was not questioned.*

The sentences that conclude the lecture are really astonishing, and one could see an anticipation of Einstein's idea of General Relativity, a geometric theory of the universe which aims to determine the metric relations from the law of gravitation, i.e. the physical mass of the bodies.

The impact that these ideas had on many cultural activities in subsequent years up to nowadays is evident, in particular on visual art. It is more subtle to sort out the full reciprocal interplay between art and mathematics, i.e. to see the influence of art in the developing of mathematical ideas. Which is however extremely useful, as David Mumford suggested [6]: *The saga of mathematics is unknown outside a narrow coterie; the high points of art are basic ingredients of a liberal education. Can we use our knowledge of the latter to open up the former?* (Notice that the title of this section has been taken from Mumford's presentation.)

As an example I like to consider the English painter William Turner (1775–1851). Generally regarded as a precursor of abstract painting, he was a master of the use of light as the ground for the representation of space. Under the influence of the theory of colors of I. Newton and W. Goethe, in his painting, the role of light in determining the colors and the structure of the space is, in my opinion, the analogous of the role of the binding forces in Riemann. Who, by the way, gave the first example of higher dimensional manifold using the space of colors.

The two paintings in Fig. 1 give an idea of what I mean.

The concept of the space as a discrete structure was developed later by visual arts, mainly by the artistic movements Impressionism and Pointillism. The two paintings in Fig. 2 are examples of these techniques.

Fig. 1 William Turner, Venice, la foce del Canal Grande, 1840; Yale Center for British Art. Light and colour, Goethe's Theory, 1843; Tate Gallery

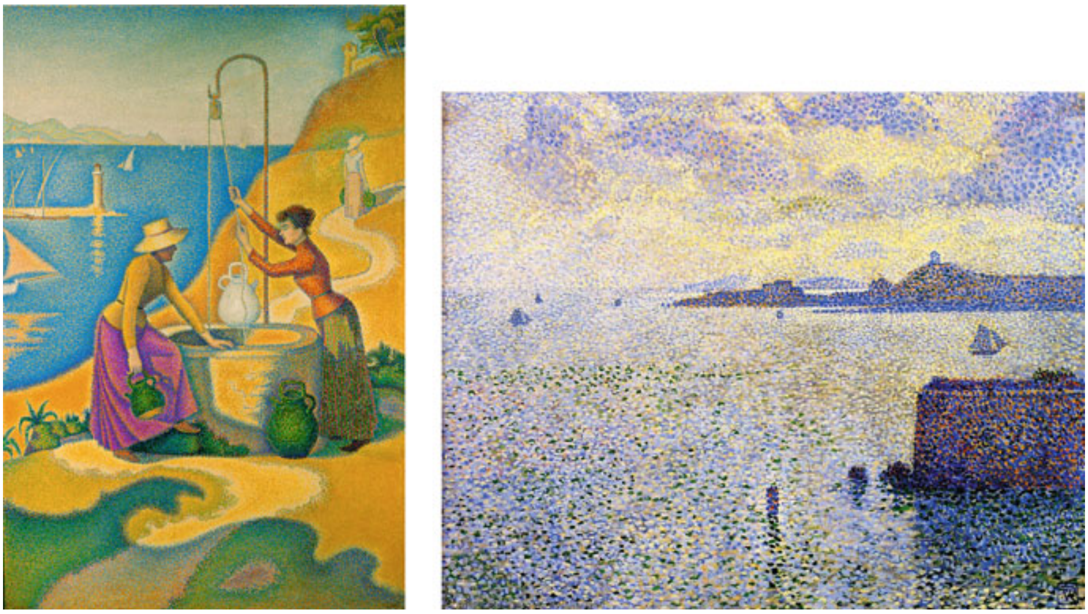

Fig. 2 Paul Signac, Femmes au Puits, 1892; Musée d'Orsay. Theo van Rysselberrghe, Sailboats and Estuary, 1892; Musée d'Orsay

2 Italian Mathematicians Contribute to Construct a European Culture

The work of Riemann gave a dramatic impulse to science, and very soon many mathematicians began to study and develop his ideas. Among them, for the purpose of this short essay, I like to point out two Italians, namely Felice Casorati (1835–1890) and Eugenio Beltrami (1835–1900). They were born in 1835, the first in Pavia and the second in Cremona, subjects of the Lombardo-Veneto Kingdom, a crown land of the Austrian Empire. In 1853, they were students at the University of Pavia, where their long-lasting friendship very likely started.

Casorati, from a middle-class family, graduated in 1856 in engineering and very soon, under the guidance of F. Brioschi, became professor of Algebra and Analytic Geometry in Pavia. Beltrami's life was much more adventurous and troubled, at least in the first part. His father, as well as his grandfather, was a painter; his participation to the Risorgimento uprisings against Austrian Army forced him to leave Italy. The young Eugenio followed the patriotic ideas of his father and as a student of Collegio Ghisleri in Pavia promoted some protests against the Rector, a pro Empire priest. He was removed from the College and, because of lack of money, he could not complete the studies and never got the "Laurea". After working for some years at the Lombardo-Veneto railway system, when Lombardia becomes a land of the kingdom of Italy he was nominated professor of Algebra and Analytic Geometry in Bologna and subsequently of Geodesic in Pisa by the vice minister of education, the mathematician F. Brioschi. In the end of his life, he was politically influent: he was Presidente of the Accademia dei Lincei and Senatore of the Italian Kingdom.

2.1 Felice Casorati

Casorati met Riemann in 1858 in Gottingen, during a famous scientific trip with Betti and Brioschi, and he was so struck by his ideas that he dedicated most of his mathematical activities to disclose them to the mathematical community. In particular, in a famous book in 1868, [4], and later in a paper of 1887, he carefully described the concept of a **Riemann surface** (Casorati called it "Riemanniana"). This is a very abstract concept which relates a geometric object as a surface with a function of a complex variable. It is a perfect interplay between geometry and analysis, between space and metaphysics. In short, to construct a Riemann surface, you take some planes, cut them transversally along prescribed segments (Riemann called these cuts *Querschnitte*), put them one over the other and glue together edges of the cuts in different planes, following a precise mathematical procedure determined by the chosen function, in order to obtain a connected surface without border.

The instructions given by Riemann and later by Casorati represent first sketches of abstract art. They aim to translate mind games into geometric objects; a main difficulty is that some of these Riemann surfaces cannot be constructed in the normal three-dimensional space, unless one introduces branch or singular points.

The instructions and a first drawing of a Riemann surface can be seen in Fig. 3; they are taken from the books by Casorati and a similar one by Carl Neumann. In Fig. 4, one can find a Riemann surface printed in a Fab Lab with a 3D printer.

There is (at least) another famous Felice Casorati (1883–1963), a painter who was the nephew of the mathematician. Referring to his uncle, he said: "My ancestors could explain the scientific order of my paintings, the rationality which pushes me towards an extreme precision, as it is for philosophers, mathematicians and some musicians". His painting *Gli scolari* (1927–1928) is represented in Fig. 5. I saw it in an exhibition at MART in Rovereto titled "Realismo Magico" (Magic Realism),

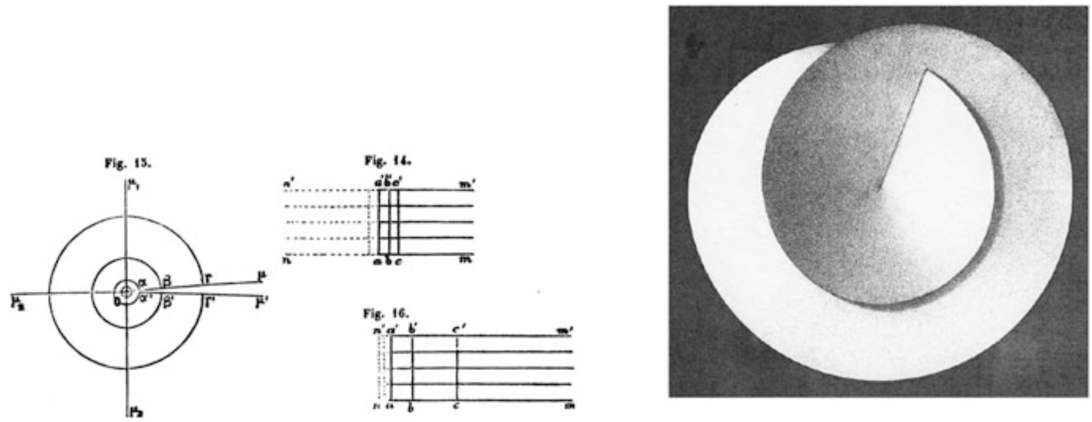

Fig. 3 Felice Casorati, Teorica delle Funzioni di Variabili Complesse, Pavia: Fusi 1868 and Carl Neumann, Vorlesungen über Riemann's Theorie der Abelschen Integralen. Leipzig: Teubner 1865

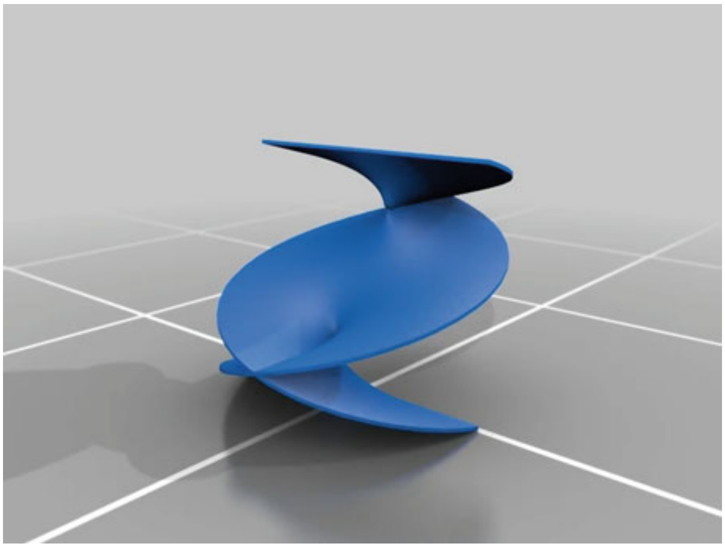

Fig. 4 3D printing of a Riemann surface

the name of a style used by some painters at the beginning of the last century to perform a realistic view of the modern world while also adding magical elements. I was impressed by the description given by one of the curators, Valerio Terraroli, who pointed out "the peculiar geometry of the floor, given through an unusual perspective which suggests a sort of unfolding". To me, this was a clear indication that both Casorati, the mathematician and the painter, were fascinated by the magic realism of the unfolding planes of Riemann surfaces.

Lucio Fontana (1899–1968) was a painter and sculptor who performed masterfully the Riemann technique of Querschnitte, although he probably never knew about Riemann. He was a founder of the artistic movement *Spatialism* and produced a series of monocrome paintings with transversal cuts denominated *Spatial Concepts or Attentes*. He described them as the "Art for the Space Age", where "the figures seem to leave the plane and enter into the space"; this seems to be a good

Fig. 5 Felice Casorati, Gli scolari, 1927; courtesy Museo di Arte Moderna di Palermo "Empedocle Restivo"

interpretation of Riemann's and Casorati's instructions. Figure 6 represents two of Fontana's spatial concepts.

Lucio Fontana realized also many paintings with a finite number of holes, possibly of different size but with a precise spatial order (for instance, displaced along a line) (see Fig. 7). This is an alternative way of describing a Riemann surface used by mathematicians, i.e. assigning a finite number of points on a plane, each with a fixed multiplicity, displayed along a curve.

I like to point out a more recent drawing by the New York visual artist Lun-Yi London Tsai. He is a mathematically trained artist able to talk with mathematicians and to visualize their scientific achievements. Figure 8 is a drawing he created in a dialogue with the mathematician Sandor Kovacs, and it represents a bunch of Riemann surfaces parametrized by another Riemann surface. It can be used to introduce a conjecture in Algebraic Geometry stated by the Russian mathematician Igor Shafarevich (1923–2017), solved and generalized in higher dimension by several mathematicians, including S. Kovacs.

2.2 Eugenio Beltrami

Eugenio Beltrami met Riemann in Pisa, a university frequently visited by the German mathematician in the last years of his life. Besides direct conversations,

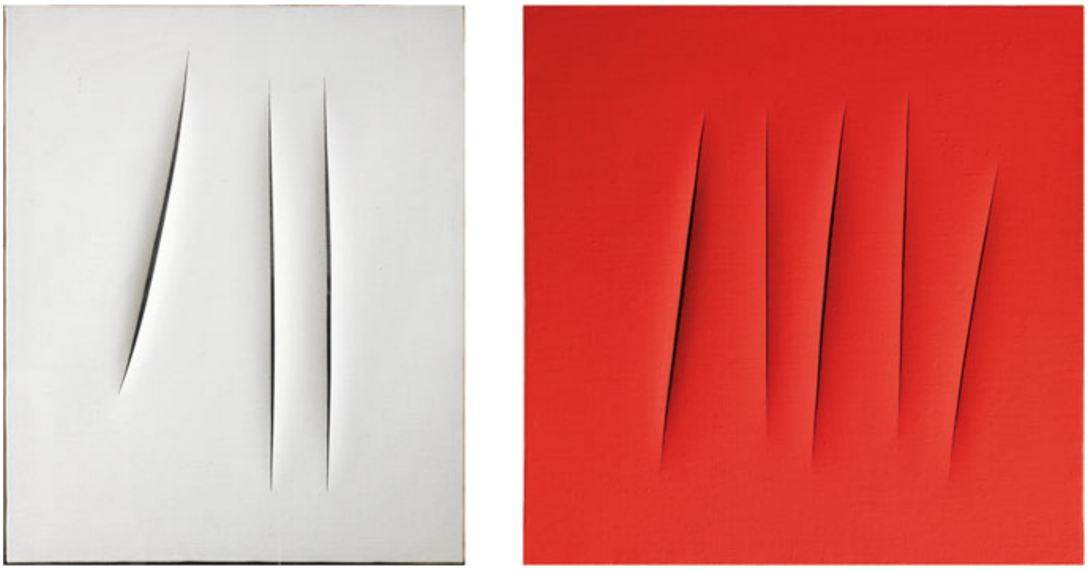

Fig. 6 Lucio Fontana, Concetto Spaziale-Attesa, 1966, courtesy Tornabuoni Art

Beltrami was a careful reader of Riemann's papers, in particular, of the fundamental lecture on geometry; together with Casorati, he ran several seminars on these subjects. In his two most famous papers, *Saggio di Interpretazione della Geometria non-Euclidea* e *Spazi di Curvatura Costanti*, both published in 1868, [2], he provided the first explicit model of non-Euclidian geometry. This is a surface of constant negative curvature, which he called *the pseudosphere*, on which all postulates of Euclidean geometry hold except the fifth: given a line (geodesic) on the surface and a point not on the line, there are infinitely many lines passing through that point and parallel to the first line. This example was a sort of holy grail which philosophers and mathematicians had been searching for hundred years; its discovery changed the way of looking at space. Beltrami gave much credit to Riemann, saying that he hoped *le mie ricerche possano aiutare l'intelligenza di alcune parti di questo profondo lavoro* (my researches can improve the understanding of some part of (Riemann) work). For those who can read Italian, Beltrami's papers are of great interest, a popularization of them can be read in my book [1].

Between 1869 and 1872, Beltrami, whose father was a miniaturist under the guidance of Francesco Hayez, constructed some paper models of his Pseudosphere, the ones in Fig. 9 are displayed at the Department of Mathematics in Pavia. The first is made of 124 pieces and was used by Casorati during the opening Lecture of the Academy year 1873–74 in Pavia.

The paper models represent a piece of the Pseudosphere, since no surface of constant negative curvature can be "embedded" in the Euclidian space, as stated by a famous theorem of Hilbert some years later. Beltrami constructed three different mathematical models of *hyperbolic geometry*; they are complete models (not a piece), and he moreover showed how to transform one into another. Two of them were later reintroduced by F. Klein and H. Poincaré, without mentioning the birthright of Beltrami.

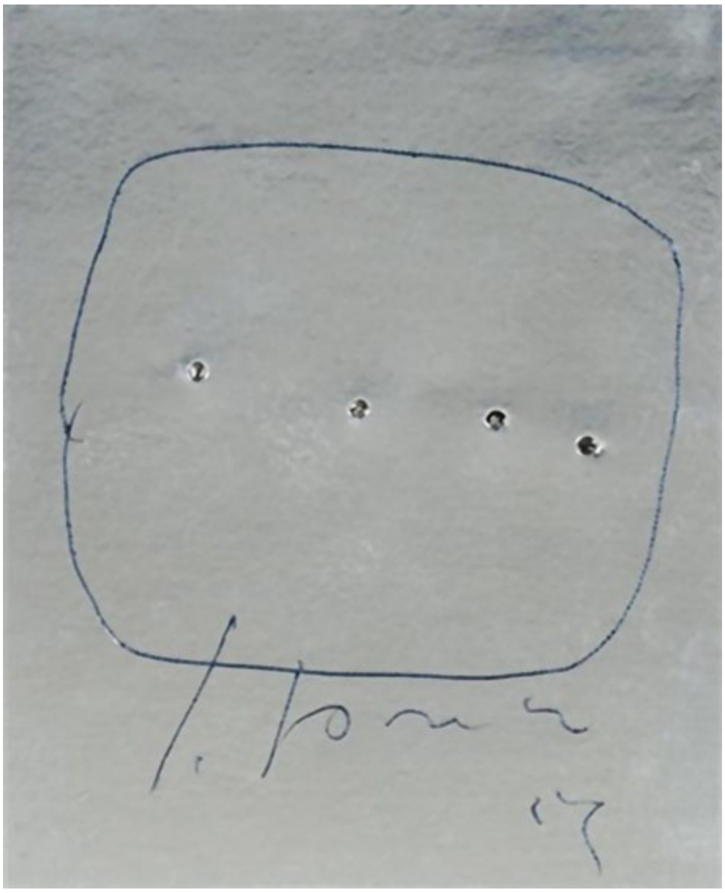

Fig. 7 Lucio Fontana, Concetto Spaziale-Buchi, 1968

Klein, who called the models *hyperbolic geometry*, was very fond of concrete models. Probably motivated by the activities of humanist colleagues who were collecting plaster copies of exemplary statues, as a professor at the Munich Institute of Technology between 1875 and 1880, he collaborated with his colleague Alexander Brill to create plaster models of mathematical surfaces; they were sold by Brill's brother Ludwig. Klein popularized a veritable zoo of models when he toured the United States in 1893, and after his visit, many American universities bought Brill's product. In 1922, Klein proudly claimed "Today, no German university any longer lacks such a collection". The qualities of seriality and eeriness made the plaster casts attractive to artists like Marcel Duchamp, Picasso, De Chirico, Carlo Carrà and Man Ray. He used the German adjective *Anschaulich*, intuitive, to describe his approach to mathematics and supported the spatial Anschauung for mathematical pedagogy. Figure 10 represents two of these surfaces reproduced by the Italian mathematician Luigi Campedelli for the Museo Nazionale di Scienza e Tecnologia, Leonardo da Vinci, Milano.

Nowadays, with a 3D printer, it is very easy to reproduce these surfaces; one can also buy them directly on the web, see, for instance, https://oliverlabs.net/math-objects/ or http://www.3dprintmath.com/, where one can buy the model of the Pseudosphere in Fig. 11.

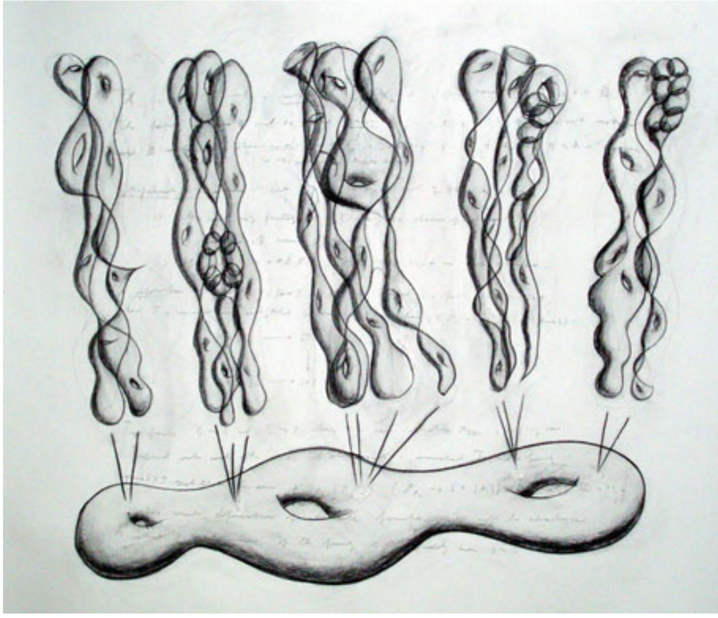

Fig. 8 Lun-Yi London Tsai, Shafarevich's Conjecture, 2007, https://www.londontsai.com/drawing

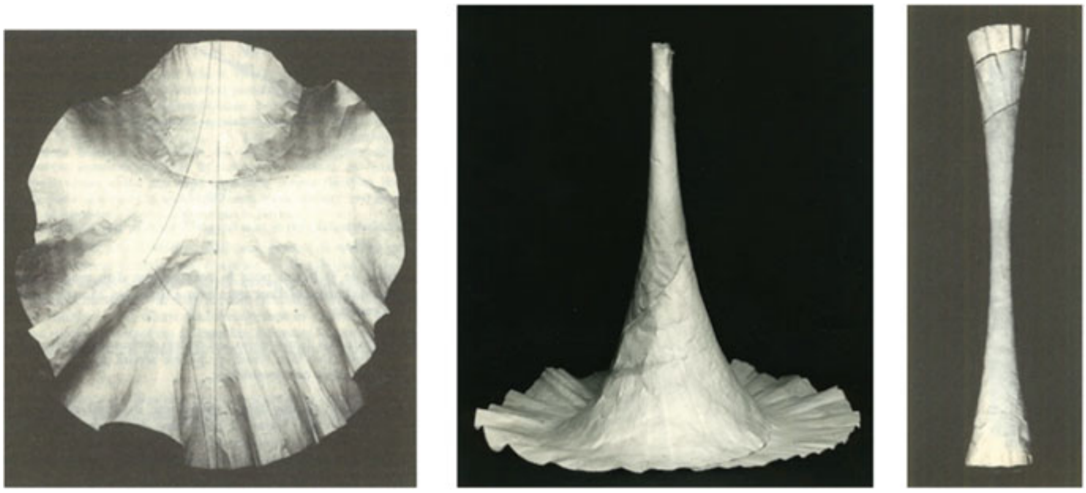

Fig. 9 Eugenio Beltrami, Pseudosphere models, 1869–1872; Department of Mathematics-Pavia

A nice hyperbolic model can be admired in the Seville city park denominated Metropol Parasole (Fig. 12, left); it resembles the shape of a mushroom or of a coral (Fig. 12, right).

One of Beltrami's models consists of a disc in the plane, and the lines are semicircles perpendicular to the border; this is the same model developed later by Klein. The Dutch graphic artist M. Cornelius Escher was fascinated by it. He was first attracted by the tessellation of the hyperbolic disk made by the mathematician H. Coxeter (Fig. 13). Out of Coxeter's drawn, Escher produced four famous engravings, titled "Circle Limits" and printed out of wooden blocks.

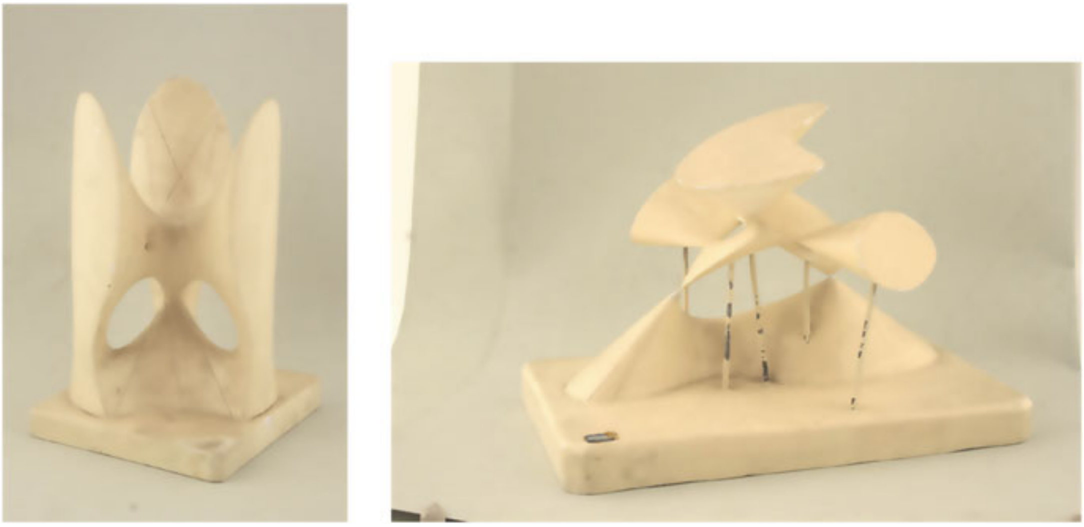

Fig. 10 Clebsch surface and Kummer surface by Campedelli, 1951; Museo Nazionale Scienza e Tecnologia, Leonardo da Vinci, Milano

Fig. 11 Henry Segerman, Tilings and curvature

The tiling proposed by Coxeter consists of hexagonal tiles, each subdivided in twelve black and white triangles; such tiling is possible also in the Euclidean space. In the hyperbolic disc, thanks to the negative curvature of the space, more tilings are allowed; in particular, one can think at a heptagonal one. Physicists Alicia J. Kollár et al. in [5], starting from a heptagonal tiling of the hyperbolic disc, constructed a finite section of it in the Euclidian plane consisting of one central heptagon and two shells of neighbouring tiles. They fabricated it in a 200 nm niobium film using photolithography, see Fig. 14. The length of the sides of the heptagon is kept equal, thanks to their curved shape, a very clever solution.

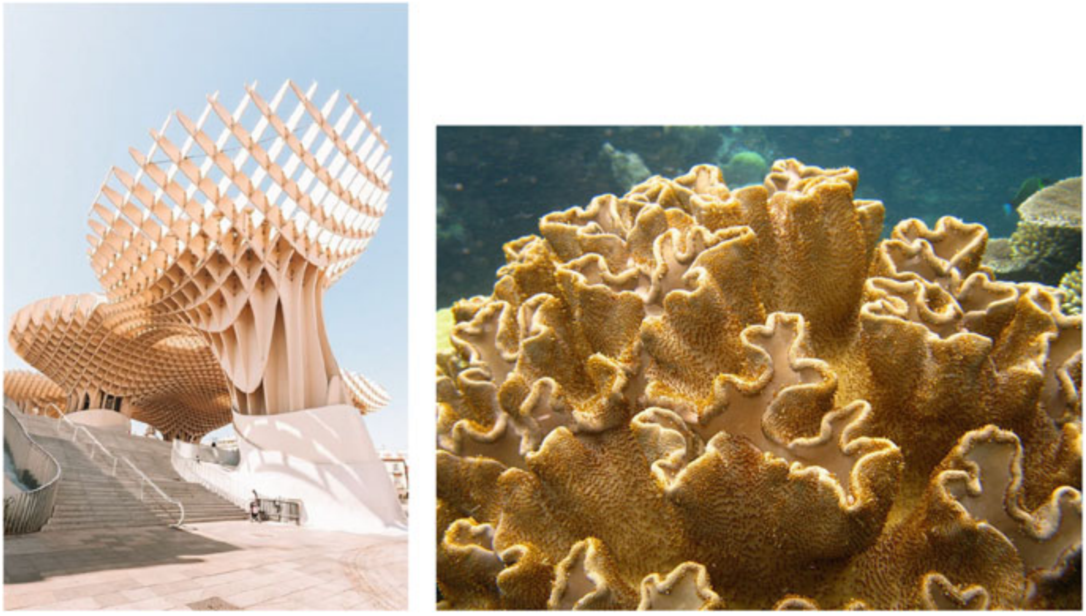

Fig. 12 Seville city park denominated Metropol Parasole, 2009. A coral

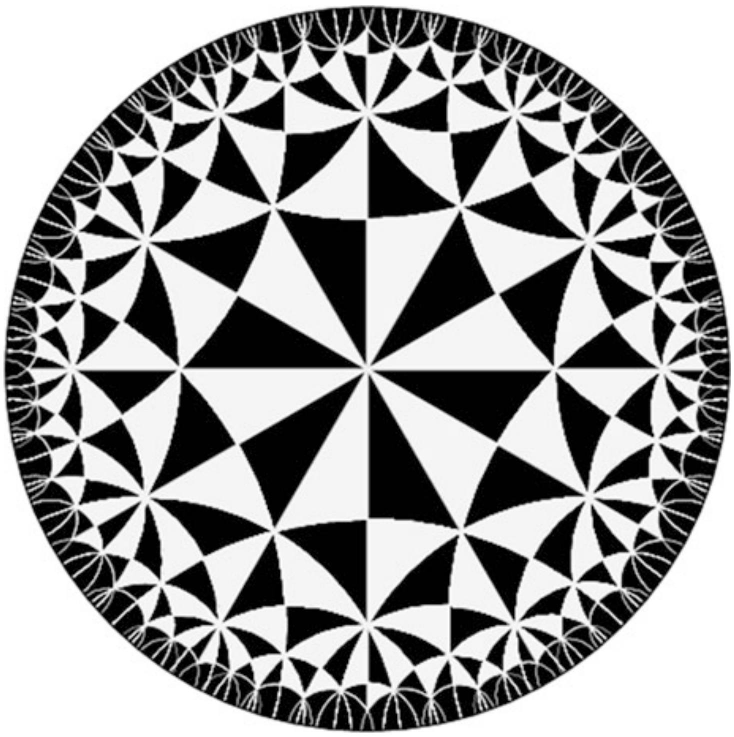

Fig. 13 H. Coxeter's tessellation of the hyperbolic disk

The funny thing is that this is not simply art; it has been proposed as a part of a future superconducting circuit, to perform quantum computation and quantum simulation.

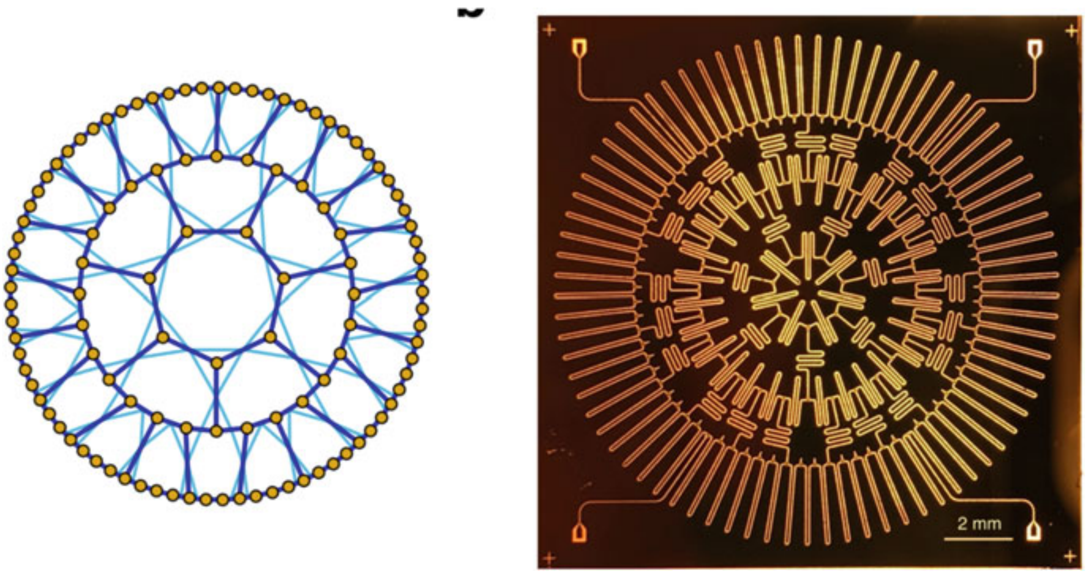

Fig. 14 Alicia J. Kollár et oth., The heptagon-kagome device, 2019; taken from [5]

Engineers and physicists, with the theoretical assistance of mathematicians, are trying to construct graphene's tissues shaped as a Pseudosphere. Graphene is an allotrope of carbon; it is one atom thick, and hence it is the closest in nature to a two-dimensional object. This is very much in the spirit of a later paper by Beltrami, [3], in which he proves that the Theory of Elasticity can be better performed on his Pseudosphere, a first taste that curved spaces are more suitable for science and art.

3 From Edgard Degas to Shigefumi Mori

The French writer and philosopher Paul Valéry (1871–1945) was first interested in Edgard Degas (1834–1917) to complete his "collection of brains", because he admired his mathematical intelligence. During their long-lasting attendance (almost 20 years), Valéry wrote the book "Degas Danse Dessin" (1938). Describing Degas's work, he wrote: *There is a huge difference between seeing something without the pencil in your hand and seeing it while drawing it. Or rather you are seeing two quite different things. Even the most familiar object becomes something else entirely, when you apply yourself to drawing it: you become aware that you did not know it-that you had never truly seen it... It dawns on me that I did not know what I knew: my best friend nose.*

I find it very pertinent to the work of mathematicians: although we are not artists, when we draw a diagram, even digitally, very often we see a different thing, maybe something we had never truly seen before.

In November 2019, the Fields medalist Shigefumi Mori gave a "Lezione Leonardesca" in Milano. During the lecture, he made a parallelism between the study of geometry and the art of painting. He started showing a drawing, Fig. 15, of

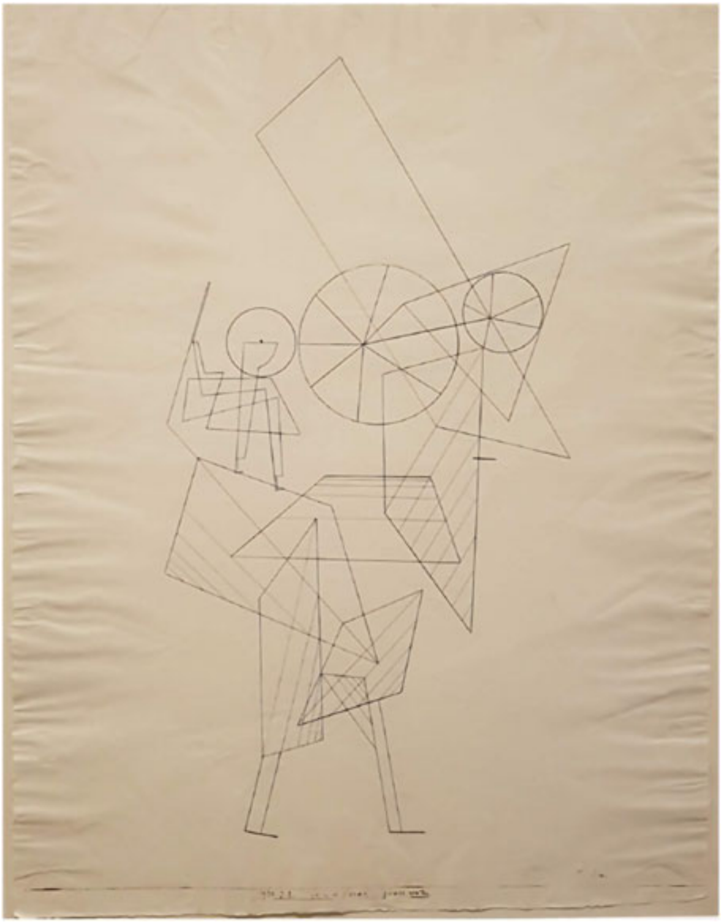

Fig. 15 Paul Klee, Steerable Grandfather (1930), Zentrum Paul Klee

Paul Klee (1879–1940), in the meantime quoting the artist: "Art does not reproduce what one can see but makes seeable what one cannot see".

S. Mori is a leading figure of the geometry movement whose Manifesto is called "Minimal Model Program". This is a scientific programme which aims to classify algebraic varieties, living in a projective space, of any dimension. To reach the classification, one needs to define a number of models and a precise procedure to connect any variety to one of these models. Working in projective spaces and in higher dimension is a very abstract matter which deals with making seeable what cannot be seen. This has many roots in the work of the Italian painters of the Early Renaissance which invented projective geometry in order to make seeable in the plane models living in the space.

In the lecture, he explained that *Geometers study figures via* ***invariants****, which correspond to the object used in Cubism painting. The use of invariants is similar to abstract paintings. An invariant is defined for each figure; unlike an artistic objects an invariant needs objectivity and reproductivity.* He himself created and contributed to describe many invariants associated with projective varieties. Among others, the *Cone of Curves*, made by all the curves which lie on a variety, was a key tool in his theory, as he explained in the lecture, see Fig. 16.

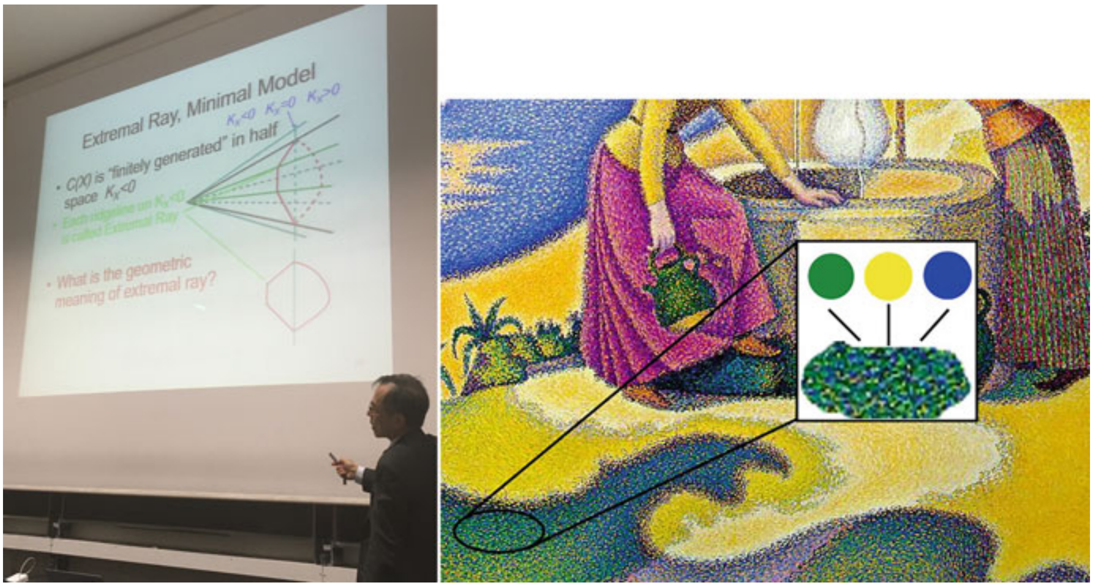

Fig. 16 Shigefumi Mori, Lezione Leonardesca, 2019; Milano. From Paul Signac1892, Wikimedia Commons

The second picture in Fig. 16 is an attempt to describe a "cone of points" associated with the Pointillism painting of Paul Signac, an artistic analogue of the geometric "cone of curves" studied by Mori and his school.

Acknowledgements I wish to express my sincere gratitude to Michele Emmer for his imaginative and fruitful work to explore the interplay between mathematics and other cultural activities.

The author was partially supported by fund of the project PRIN 2017 "Moduli Theory and Birational Classification" (Italy).

References

1. Andreatta, M.: La Forma delle Cose, l'alfabeto della Geometria. Il Mulino, Bologna (2019)
2. Beltrami, E.: Saggio di Interpretazione della Geometria non-Euclidea. G. Mat. **VI**, 284–322 (1868). Spazi di Curvatura Costanti. Ann. Mat. Pura Appl. **Serie II, II** , 232–255 (1868)
3. Beltrami, E.: Sulle Equazioni Generali dell'Elasticità. Ann. Mat. Pura Appl. **Serie II, X**, 188–211 (1882)
4. Casorati, F.: Teorica delle Funzioni di Variabile Complessa. Tipografia Fusi, Pavia (1868)
5. Kollár, A.J., Mattias Fitzpatrick, M., Houck, A.A.: Hyperbolic lattices in circuit quantum electrodynamics. Nature **571**, 45–50 (2019)
6. Mumford, D.: Art, Mathematics and the Zeitgeist: parallels between the two most international disciplines. In: Festival of Mathematics. Rome (2008). Available via D. Mumford's Archive. https://www.dam.brown.edu/people/mumford/beyond/papers/2008f-ArtMathZeitgeist-talk.pdf. Cited February 2021