Fano's Last Fano

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Introduction

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We consider normal projective varieties X defined over $\mathbb{C}$.
If n is the dimension of X we sometime call X and *n-fold*;
we denote by K_X the *canonical sheaf*.
We assume to have good singularities such that K_X, or a multiple of it, is a line bundle (a Cartier divisor).

Fano's 3folds

Let $X \subset \mathbb{P}^N$ be a projective 3-fold such that for general hyperplanes H_1, H_2 the curve $\Gamma := X \cap H_1 \cap H_2$ is canonically embedded (i.e. K_Γ embeds Γ).

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This is the case if and only if the the linear system, $|-K_X]$, embeds X as a 3-fold of degree $2g-2$ into a projective space of dimension $g+1$, $X := \mathbf{X_3^{2g-2}} \subset \mathbb{P}^{\mathbf{g+1}}$, where $g = g(\Gamma)$ is the genus of Γ.

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Example: the quartic 3-fold in $\mathbb{P}^4$, $X_3^4 \subset \mathbb{P}^4$.

Fano Varieties

Definition

A smooth projective variety X is called a *Fano manifold* if $-K_X$ is ample.

The *index of* X is defined as the greatest integer which divides $-K_X$, that is the greatest r such that $-K_X = rL$ for a line bundle L.

If $Pic(X) = \mathbb{Z}$ then X is called a *Fano manifold of the first species* or a *prime Fano manifold*.

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Proposition

Let $X \subset \mathbb{P}^N$ be a projective n-fold and let $H := \mathcal{O}_{\mathbb{P}^N}(1)_{|X}$ be the hyperplane bundle. Assume that for general hyperplanes $H_1, H_2, \ldots, H_{n-1} \in |H|$ the curve $\Gamma := H_1 \cap H_2 \cap \ldots \cap H_{n-1}$ is a canonically embedded curve of genus g. Then the anticanonical bundle is linearly equivalent to $(n-2)$ times the hyperplane bundle, i.e. $-K_X = (n-2)H$.

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Su una particolare varietà a tre dimensioni a curve-sezioni canoniche [1]

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L. Roth cited the paper at page 93 of his book *Algebraic Threefolds* (1955) saying that Fano examined *a particular fourfold of the third species* ...; probably Roth read the paper too quickly and did not realize that Fano was actually searching for a 3-fold and not (only) for a 4-fold.

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It is not *prime*, i.e. it has Picard rank 2. Therefore it is not isomorphic to either the Iskovskikh or the Mukai example and it should be searched in the Mori-Mukai classification.

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Fano's paper

Geometria algebrica. — *Su una particolare varietà a tre dimensioni a curve–sezioni canoniche.* Nota (*) del Socio GINO FANO.

1. Ho incontrato recentemente una varietà a tre dimensioni a curve-sezioni canoniche, che naturalmente appartiene alla serie delle M_3^{2p-2} di S_{p+1} (qui $p = 12$), oggetto di mie ricerche in quest'ultimo periodo (1), ma non ha finora richiamata particolare attenzione. Ne darò qui un breve cenno.

Consideriamo nello spazio S_5 una rigata razionale normale R^4 (non cono), che per semplicità supponiamo del tipo più generale, cioè con ∞^1 coniche direttrici irriducibili; e con essa la varietà ∞^4 delle sue corde. Quale ne è l'immagine M_4 nella Grassmanniana M_8^{14} di S_{14} (2) delle rette di S_5 (3)?

Determiniamo anzitutto l'ordine di questa M_4, ad esempio l'ordine della superficie sua intersezione con un S_{12}, vale a dire della ∞^2 di rette comune alla ∞^4 suddetta e a due complessi lineari. Valendoci di due complessi costituiti risp. dalle rette incidenti a due S_3, questi ultimi contenuti in un $S_4 \equiv \sigma$ e aventi perciò a comune un piano π, la ∞^2 di rette in parola si spezzerà nei due sistemi delle corde di R^4 contenute in σ e di quelle incidenti al piano π. Le prime sono le ∞^2 corde di una C^4 razionale normale, e nella Grassmanniana delle rette di σ hanno per immagine una superficie φ^9 di S_9 di Del Pezzo (4). Della seconda ∞^2 prendiamo l'intersezione con un ulteriore complesso lineare, anche con un $S_3 \equiv \tau$ direttore incontrante π in una retta. Si ha una rigata composta di una parte luogo delle corde di R^4 contenute nello spazio $S_4 \equiv \tau\pi$ e incidenti a π, la cui imma-

(*) Presentata nella seduta dell'8 gennaio 1949.

(1) Più specialmente nella Memoria: *Sulle varietà algebriche a tre dimensioni a curve-sezioni canoniche.* «Mem. Acc. d'Italia», classe sc. fis., vol. VIII (1937), n. 2.

gine è sezione iperpiana di altra φ^9 di Del Pezzo; e di una seconda parte luogo delle corde incidenti alla retta $\tau\pi$. Quest'ultima rigata è di 4° ordine, avendo la retta $\tau\pi$ come direttrice semplice, e 3 generatrici in ogni S_4 per essa (poichè la proiezione della rigata dalla retta $\tau\pi$ ha una cubica doppia). Complessivamente la superficie immagine delle corde di R^4 appoggiate a un piano è dunque di ordine $9 + 4 = 13$ (5); e la M_4 immagine del sistema di tutte le corde di R^4 è di ordine $9 + 13 = 22$ (6). Le due superficie φ^9 e F^{13}, costituenti insieme una sezione superficiale della M_4^{22}, hanno a comune una curva sezione iperpiana della φ^9 (collo spazio σ), perciò ellittica, di ordine 9; la M_4^{22} ha quindi superficie–sezioni di genere uno, e curve–sezioni canoniche di genere 12 (appunto $= 1 + 3 + 9 - 1$). Le sezioni iperpiane della M_4^{22} sono pertanto M_3^{22} di S_{13}, corrispondenti al tipo generale M_3^{2p-2} di S_{p+3} per $p = 12$, e razionali (come risulterà pure dai sistemi lineari di superficie che vi sono contenuti). Indicheremo d'ora in poi questa varietà con μ_3^{22}, o semplicemente μ; essa è l'immagine del sistema ∞^3 di rette Σ intersezione della ∞^4 delle corde di R^4 con un complesso lineare **K** (che si supporrà per ora del tipo più generale, e in posizione generica rispetto a R^4).

...to prove...

Fano considered R^4, the ruled surface image of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^5$ by the complete linear system $|(1,2)|$. It is rational, it has degree 4 and its general hyperplane section is a smooth rational curve of degree 4.

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The variety M_4 is defined by Fano as the subset of the Grassmannian of lines in $\mathbb{P}^5$ (embedded via the Plücker embedding as $M_8^{14} \subset \mathbb{P}^{14}$) given by the *chords* of R^4.
Since we are looking for a complete variety, we need to interpret the word "corde" in a broad sense, that is secant and tangent lines.

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Fano had to prove the following:

Proposition

- $M_4 \subset M_8^{14} \subset \mathbb{P}^{14}$ *is an irreducible smooth variety of dimension* 4
- $\deg M_4 = 22$
- *The sectional genus of* M_4^{22} *is equal to* 12

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He described two surfaces in M_3 (not numerically equivalent).

Computing the degree

Take two special hyperplanes sections in M_8^{14} given by the lines intersecting two linear subspaces of dimension 3 in $\mathbb{P}^5$ which are in "special position": i.e. they intersect along a plane π or equivalently that both are contained in a hyperplane $\sigma \subset \mathbb{P}^5$.

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He notices that the lines in the intersection of the two hyperplanes in M_8^{14} are exactly the lines contained in σ and the lines intersecting π. Denote with S^σ the subvariety of M_4 of the lines contained in σ and with S_π the subvariety of lines intersecting π.

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Lemma

$$\deg M_4 = \deg S^\sigma + \deg S_\pi$$

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Proof.

$R^4 \cap \sigma = C^4$ is a rational normal curve of degree 4 in $\sigma = \mathbb{P}^4$. Lines contained in $\mathbb{P}^4 \subset \mathbb{P}^5$ are mapped by the Plücker embedding into $\mathbb{P}^9$. Therefore S^σ is the image of $S^2(C^4) = \mathbb{P}^2 \to \mathbb{P}^9$ which maps a pair (p, q) to the secant $\overline{pq}$.

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Consider the hyperplane section given by the secants of C^4 intersecting a fixed general plane $\pi \subset \sigma$ and its pullback $H \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d, d)|$ to $\mathbb{P}^1 \times \mathbb{P}^1$. Take a general point $p \in C^4$ not contained in π.
d is equal the number of points $q \in C^4$ such that $\overline{pq}$ is a secant to C^4 intersecting π, i.e. it is equals the number of secants through a general point $p \in C^4$ intersecting π.
Take the projection $f_p \colon \sigma \dashrightarrow \mathbb{P}^3$. The secants through p intersecting π are projected to the points of the plane $f_p(\pi)$ intersecting the rational normal cubic $f_p(C^4)$, so there are exactly 3 of them: $d = 3$.

Degree of S_π and finally of M_4

Let $\tau \subset \mathbb{P}^5$ a special codimension two space that intersects π in a line and consider the special hyperplane section of S_π given by the lines that incide τ.

This curve has two irreducible components: the secants in S_π

- contained in the $\mathbb{P}^4$ generated by τ and π and intersecting π, $C_\pi^{\langle \tau, \pi \rangle}$
- those intersecting the line $\tau \cap \pi$, $C_{\tau \cap \pi}$.

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For general choice of π, τ

$$\deg S_\pi = \deg C_\pi^{\langle \tau, \pi \rangle} + C_{\tau \cap \pi} = 9 + 4 = 13$$

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Lemma

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Theorem

$$\deg M_4 = \deg S^\sigma + \deg S_\pi = 9 + 13 = 22$$

M_4 is Fano

Takes a hyperplane section of $S^\sigma \cup S_\pi$ to compute the sectional genus of M_4^{22}.
Since S^σ is the Del Pezzo of degree 9, its general hyperplane section is a smooth plane cubic, which has genus 1.
$C_\pi^{\langle\tau,\pi\rangle} \cup C_{\tau\cap\pi}$ is a reducible hyperplane section of S_π, formed by two smooth curves of respective genus 0 and 1 intersecting in 3 points: it follows that the general hyperplane section is a smooth curve of genus $0+1+3-1=3$.
The intersection of S^σ and S_π is a hyperplane section of S^σ, a curve of degree 9. So the two curves obtained cutting S^σ and S_π with a general hyperplane intersect in 9 points.
The sectional genus of M_4^{22}, which is the genus of a hyperplane section of $S^\sigma \cup S_\pi$, is equal to $1+3+9-1=12$.

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The sectional genus of M_4^{22}, which is the genus of a hyperplane section of $S^\sigma \cup S_\pi$, is equal to $1 + 3 + 9 - 1 = 12$.

Proposition

A general curve section of M_4^{22} is therefore a non-degenerate smooth curve of genus 12 in $\mathbb{P}^{14-3=11}$ of degree 22; by Riemann-Roch this is a canonical curve, i.e. it is embedded by its complete canonical system.

Generalized Fano's construction

Let S be any smooth projective variety and consider the Hilbert Scheme which parametrizes its zero dimensional subschemes of length 2, $S^{[2]}$. Let $\varphi : S^{[2]} \to S^{(2)}$ be the Hillb to Chow map; it contracts a divisor $D \subset S^{[2]}$ to a surface in $S^{(2)}$.

If the irregularity of S is zero then $S^{[2]}$ is smooth (Fogarty-1968) and φ is a crepant contraction.

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Choose an embedding $S \hookrightarrow \mathbb{P}^N$ and consider the natural map $S^{[2]} \to G(1, N)$ associating to each subscheme of length 2 of S the unique line containing its image in $\mathbb{P}^N$. Compose further with the Plücker embedding of the Grassmannian, $Gr(1, N) \to \mathbb{P}^{\frac{(N+1)N}{2}-1}$.

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By the classical so called trisecant lemma, if $N \geq 4$ the total map

$$S^{[2]} \to \mathbb{P}^{\frac{(N+1)N}{2}-1}$$

is a birational map (onto its image).

The image is **the variety of the lines that are secants or tangents to** $S \subset \mathbb{P}^N$.

The case of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[2]}$

Proposition

Let $\mathcal{H}$ *be the Hilbert Scheme* $(\mathbb{P}^1 \times \mathbb{P}^1)^{[2]}$

a) $\mathcal{H}$ *is a smooth projective variety of dimension* 4

b) $Pic(\mathcal{H}) = \mathbb{Z}(H_1^{[2]}) \oplus \ Z(H_2^{[2]}) \oplus \mathbb{Z}(B/2)$

c) $Nef(\mathcal{H})$ *is the simplicial cone:* $< H_1^{[2]}, H_2^{[2]}, H_1^{[2]} + H_2^{[2]} - (B/2) >$

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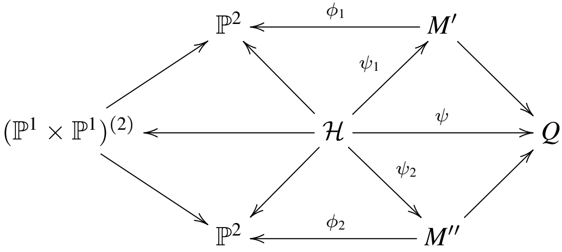
The diagram represents the maps associated to the nef bundles on $\mathcal{H}$.



Take the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, given by the complete linear system $(1,1)$, as a smooth quadric surface $Q_2 \subset \mathbb{P}^3$.
Note that that the secant lines fill up the whole Grassmannian $G(1,3)$, since every line in $\mathbb{P}^3$ is secant to any quadric surface. The Plücker embedding maps $G(1,3)$ into a (Klein) quadric $4-$fold Q_4 in $\mathbb{P}^5$.
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All zero cycles on a fiber of the ruling are contracted to a point; therefore two divisors, D_1 and D_2 in $\mathcal{H}$, are contracted to two conics, $C_1, C_2 \subset Q_4$, which describe in the Grassmannian the lines in the ruling.

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All non zero dimensional fibers of ψ have the same dimension (and are isomorphic to $\mathbb{P}^2$).. By a general result, [A-Wisniewski], $\psi : \mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1) \to Q_4 \subset \mathbb{P}^5$ is the blow up of the quadric $Q_4 \subset \mathbb{P}^5$ along two disjoint smooth conics, C_1, C_2.

Fano's Construction revisited

Embed $S = \mathbb{P}^1 \times \mathbb{P}^1$ by the linear system $(1, 2)$ as the normal rational scroll of degree 4, $R^4 \subset \mathbb{P}^5$.

The birational map $\psi_1 : \mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1) \to M'$ **is, by construction, exactly the one in Fano's paper**: $M' = M_4 \subset \mathbb{P}^{14}$.

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Fano's Construction revisited

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The other divisor $E := D_1$ remains isomorphically equal in M_4 and it can be contracted as a smooth blow-down to the curve $C_1 \subset Q_4 \subset \mathbb{P}^5$, $\nu : M_4 \to Q_4$.

A new description

Let $\nu\colon M_4 \to Q_4$ be the blow-up of a smooth conic $C_1 \subset Q_4 \subset \mathbb{P}^5$ (not contained in any plane $\mathbb{P}^2 \subset Q_4 \subset \mathbb{P}^5$).

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Let H be the hyperplane bundle in $\mathbb{P}^5$; the formula for the canonical bundle of the blow up gives

$$-K_{M_4} = \nu^*(4H) - 2E = 2(\nu^*(2H) - E).$$

The line bundle $\mathcal{L} := \nu^*(2H) - E$ is very ample; it embeds M_4 into $\mathbb{P}^{14}$ as a Fano manifolds of index 2 and genus 12.

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In the classification obtained by Mukai of Fano 4- folds of index 2 (coindex 3 in Mukai' notation) one can find M_4, given as the blow-up of a four dimensional quadric along a conic, as the only one of genus 12 (Example 2). The classification was based on Conjecture (ES) which was later proved by Mella.

Mori-Mukai classification

A general hyperplane section in $\mathcal{L} = \nu^*(2H) - E$ is a Fano $3-$fold, which we denote as Fano did with M_3^{22}. Since, as we have seen above, $\mathcal{L}$ embeds M_4 as the image of Q_4 by the rational map given by the quadric hypersurfaces through a general (=not contained in a plane) conic in Q_4, the hyperplane section M_3^{22} is obtained blowing up the conic in the intersection of Q_4 with another quadric containg the conic.

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This proofs that the M_3^{22}, Fano's last Fano, is the number 16 in the Mori-Mukai list of Fano 3-folds with Picard number 2. In fact they describe this case as the blow up along a conic of a complete intersection of two quadrics in $\mathbb{P}^5$.