Fano's Last Fano

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Advertising in Rumenia ....

Introduction

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A large number of mathematicians, often organized in counterposed schools, in the last 50 years, starting from Fano's results, constructed theories which are among the most spectacular achievements of contemporary mathematics.

Notation

In the lecture we consider normal projective varieties X defined over $\mathbb{C}$.
If n is the dimension of X we sometime call X and *n-fold*;
we denote by K_X the *canonical sheaf*.
We assume to have good singularities such that K_X, or a multiple of it, is a line bundle (a Cartier divisor).

Fano's 3folds

Let $X \subset \mathbb{P}^N$ be a projective 3-fold such that for general hyperplanes H_1, H_2 the curve $\Gamma := X \cap H_1 \cap H_2$ is canonically embedded (i.e. K_Γ embeds Γ).

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This is the case if and only if the the linear system, $|-K_X]$, embeds X as a 3-fold of degree $2g-2$ into a projective space of dimension $g+1$, $X := X_3^{2g-2} \subset \mathbb{P}^{g+1}$, where $g = g(\Gamma)$ is the genus of Γ.

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Example: the quartic 3-fold in $\mathbb{P}^4$, $X_3^4 \subset \mathbb{P}^4$.

Fano's 3folds

Fano noticed that for such varieties the following invariants are zero:

- $P_m(X) = h^0(X, mK_X) = 0$ for all $m \geq 1$ (m-th plurigenera) (we say that X has Kodaira dimension minus infinity: $k(X) = -\infty$)
- $h^i(\mathcal{O}_X) = 0$ for all positive i (in particular the irregularity $q(X) = h^1(X, \mathcal{O}_X)$ is zero).

Varieties satisfying these two conditions were called by him Varietà algebriche a tre dimensioni aventi tutti i generi nulli.

Non rational 3-folds

Fano had the insight that among this class of varieties there are varieties which are non-rational, in spite of the fact that they have all plurigenera and irregularity equal to zero; they would provide a counterexample to a Castelnuovo type rationality criteria for 3-folds.
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The first proof of the non rationality of all $X_3^4 \subset \mathbb{P}^4$ is the celebrated Iskovskikh and Manin's. B. Segre has constructed some unirational $X_3^4 \subset \mathbb{P}^4$, therefore they represents counterexamples to Lüroth problem in dimension 3 (as well as to a Castelnuovo type rationality criteria).

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In the same period Clemens and Griffiths proved the non-rationality of the cubic 3-fold in $\mathbb{P}^4$.
Both papers gave rise to subsequent deep results and theories aimed to determine the rationality or not of Fano varieties.

Fano Varieties

Definition

A smooth projective variety X is called a *Fano manifold* if $-K_X$ is ample.

The *index of* X is defined as the greatest integer which divides $-K_X$, that is the greatest r such that $-K_X = rL$ for a line bundle L.

If $Pic(X) = \mathbb{Z}$ then X is called a *Fano manifold of the first species* or a *prime Fano manifold*.

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Proposition

Let $X \subset \mathbb{P}^N$ be a projective n-fold and let $H := \mathcal{O}_{\mathbb{P}^N}(1)_{|X}$ be the hyperplane bundle. Assume that for general hyperplanes $H_1, H_2, \ldots, H_{n-1} \in |H|$ the curve $\Gamma := H_1 \cap H_2 \cap \ldots \cap H_{n-1}$ is a canonically embedded curve of genus g. Then the anticanonical bundle is linearly equivalent to $(n-2)$ times the hyperplane bundle, i.e. $-K_X = (n-2)H$.

Classification and MMP

Fano Varieties are the building blocks (atoms) of the classification of projective varieties: the Minimal Model Program (MMP), a program aimed to classify projective varieties.

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He proved that $3 \leq g \leq 12$ and $g \neq 11$ and for every such g he gave a satisfactory description of the associated Fano variety.
$X_3^{22} \subset \mathbb{P}^{13}$ (omitted by Fano and later by Roth): the double projection from a line, $\pi_{2Z} : X_3^{22} \cdots > W \subset \mathbb{P}^6$, goes into W, a Fano 3-fold of index 2, degree 5, $Pic(W) = \mathbb{Z}$ and one singular point. X_3^{22} is rational.

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S. Mori and S. Mukai (1981): classified Fano 3-fold with $\rho(X) \geq 2$. At the Fano Conference in Torino (2002) they announced they have omitted the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of tridegree $(1, 1, 3)$.

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Several projects aiming to classify singular Fano varieties in dimension $3, 4$ and 5. It is estimated that 500 million shapes can be defined algebraically in four dimensions, and a few thousand more in the fifth.

Relative Case

Definition

Let $f : X \to Y$ be a contraction (divisorial, small or of fiber type), X with mild singularities; f is called a *Fano-Mori contraction* if $-K_X$ is f-ample.

If $Pic(X/Y) = \mathbb{Z}$ then X is called a *elementary Fano-Mori contraction*; if $-K_X \sim_f rL$, r is called the *nef value of f*.

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Classification of Fano-Mori contractions: Mori, Kawamata, Kollar, ... A.-Wisniewski, ... A.-Tasin (the case divisorial of nef value $> n-3$.)

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It is not *prime*, i.e. it has Picard rank 2. Therefore it is not isomorphic to either the Iskovskikh or the Mukai example and it should be searched in the Mori-Mukai classification.

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Geometria algebrica. — *Su una particolare varietà a tre dimensioni a curve-sezioni canoniche.* Nota (*) del Socio GINO FANO.

1. Ho incontrato recentemente una varietà a tre dimensioni a curve-sezioni canoniche, che naturalmente appartiene alla serie delle M_3^{2p-2} di S_{p+1} (qui $p = 12$), oggetto di mie ricerche in quest'ultimo periodo (1), ma non ha finora richiamata particolare attenzione. Ne darò qui un breve cenno.

Consideriamo nello spazio S_5 una rigata razionale normale R^4 (non cono), che per semplicità supponiamo del tipo più generale, cioè con ∞^1 coniche direttrici irriducibili; e con essa la varietà ∞^4 delle sue corde. Quale ne è l'immagine M_4 nella Grassmanniana M_8^{14} di S_{14} (2) delle rette di S_5 (3)?

Determiniamo anzitutto l'ordine di questa M_4, ad esempio l'ordine della superficie sua intersezione con un S_{12}, vale a dire della ∞^2 di rette comune alla ∞^4 suddetta e a due complessi lineari. Valendoci di due complessi costituiti risp. dalle rette incidenti a due S_3, questi ultimi contenuti in un $S_4 \equiv \sigma$ e aventi perciò a comune un piano π, la ∞^2 di rette in parola si spezzerà nei due sistemi delle corde di R^4 contenute in σ e di quelle incidenti al piano π. Le prime sono le ∞^2 corde di una C^4 razionale normale, e nella Grassmanniana delle rette di σ hanno per immagine una superficie φ^9 di S_9 di Del Pezzo (4). Della seconda ∞^2 prendiamo l'intersezione con un ulteriore complesso lineare, anche con un $S_3 \equiv \tau$ direttore incontrante π in una retta. Si ha una rigata composta di una parte luogo delle corde di R^4 contenute nello spazio $S_4 \equiv \tau\pi$ e incidenti a π, la cui imma-

(*) Presentata nella seduta dell'8 gennaio 1949.

(1) Più specialmente nella Memoria: *Sulle varietà algebriche a tre dimensioni a curve-sezioni canoniche.* «Mem. Acc. d'Italia», classe sc. fis., vol. VIII (1937), n. 2.

Fano's paper

152 Lincei – Rend. Sc. fis. mat. e nat. – Vol. VI – febbraio 1949.

gine è sezione iperpiana di altra φ^9 di Del Pezzo; e di una seconda parte luogo delle corde incidenti alla retta $\tau\pi$. Quest'ultima rigata è di 4° ordine, avendo la retta $\tau\pi$ come direttrice semplice, e 3 generatrici in ogni S_4 per essa (poichè la proiezione della rigata dalla retta $\tau\pi$ ha una cubica doppia). Complessivamente la superficie immagine delle corde di R^4 appoggiate a un piano è dunque di ordine $9 + 4 = 13$ [5]; e la M_4 immagine del sistema di tutte le corde di R^4 è di ordine $9 + 13 = 22$ [6]. Le due superficie φ^9 e F^{13}, costituenti insieme una sezione superficiale della M_4^{22}, hanno a comune una curva sezione iperpiana della φ^9 (collo spazio σ), perciò ellittica, di ordine 9; la M_4^{22} ha quindi superficie–sezioni di genere uno, e curve–sezioni canoniche di genere 12 (appunto $= 1 + 3 + 9 - 1$). Le sezioni iperpiane della M_4^{22} sono pertanto M_3^{22} di S_{13}, corrispondenti al tipo generale M_3^{2p-2} di S_{p+3} per $p = 12$, e razionali (come risulterà pure dai sistemi lineari di superficie che vi sono contenuti). Indicheremo d'ora in poi questa varietà con μ_3^{22}, o semplicemente μ; essa è l'immagine del sistema ∞^3 di rette Σ intersezione della ∞^4 delle corde di R^4 con un complesso lineare **K** (che si supporrà per ora del tipo più generale, e in posizione generica rispetto a R^4).

...to prove...

R^4 is the ruled surface image of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^5$ by the complete linear system $|(1,2)|$. It is rational, it has degree 4 and its general hyperplane section is a smooth rational curve of degree 4.

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The variety M_4 is defined by Fano as the subset of the Grassmannian of lines in $\mathbb{P}^5$ (embedded via the Plücker embedding as $M_8^{14} \subset \mathbb{P}^{14}$) given by the *chords* of R^4.
Since we are looking for a complete variety, we need to interpret the word "corde" in a broad sense, that is secant and tangent lines.

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Proposition

The above described $M_4 \subset M_8^{14} \subset \mathbb{P}^{14}$ is an irreducible smooth variety of dimension 4.

Computing the degree

Take two special hyperplanes sections in M_8^{14} given by the lines intersecting two linear subspaces of dimension 3 in $\mathbb{P}^5$ which are in "special position": i.e. they intersect along a plane π or equivalently that both are contained in a hyperplane $\sigma \subset \mathbb{P}^5$.

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Denote with S^σ the subvariety of M_4 of the lines contained in σ and with S_π the subvariety of lines intersecting π.

Lemma

$$\deg M_4 = \deg S^\sigma + \deg S_\pi$$

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Proof.

$R^4 \cap \sigma = C^4$ is a rational normal curve of degree 4 in $\sigma = \mathbb{P}^4$. Lines contained in $\mathbb{P}^4 \subset \mathbb{P}^5$ are mapped by the Plücker embedding into $\mathbb{P}^9$. Therefore S^σ is the image of $S^2(C^4) = \mathbb{P}^2 \to \mathbb{P}^9$ which maps a pair (p, q) to the secant $\overline{pq}$.

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Consider the hyperplane section given by the secants of C^4 intersecting a fixed general plane $\pi \subset \sigma$ and its pullback $H \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d, d)|$ to $\mathbb{P}^1 \times \mathbb{P}^1$. Take a general point $p \in C^4$ not contained in π.
d is equal the number of points $q \in C^4$ such that $\overline{pq}$ is a secant to C^4 intersecting π, i.e. it is equals the number of secants through a general point $p \in C^4$ intersecting π.
Take the projection $f_p : \sigma \dashrightarrow \mathbb{P}^3$. The secants through p intersecting π are projected to the points of the plane $f_p(\pi)$ intersecting the rational normal cubic $f_p(C^4)$, so there are exactly 3 of them: $d = 3$.

Degree of S_π and finally of M_4

Let $\tau \subset \mathbb{P}^5$ a special codimension two space that intersects π in a line and consider the special hyperplane section of S_π given by the lines that incide τ.
This curve has two irreducible components: the secants in S_π contained in the unique $\mathbb{P}^4$ generated by τ and π and intersecting π, $C_\pi^{\langle\tau,\pi\rangle}$, and those intersecting the line $\tau \cap \pi$, $C_{\tau\cap\pi}$.

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For general choice of π, τ

$$\deg S_\pi = \deg C_\pi^{\langle \tau, \pi \rangle} + C_{\tau \cap \pi} = 9 + 4 = 13$$

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Theorem

$$\deg M_4 = \deg S^\sigma + \deg S_\pi = 9 + 13 = 22$$

M_4 is Fano

Takes a hyperplane section of $S^\sigma \cup S_\pi$ to compute the sectional genus of M_4^{22}.
Since S^σ is the Del Pezzo of degree 9, its general hyperplane section is a smooth plane cubic, which has genus 1.
$C_\pi^{\langle\tau,\pi\rangle} \cup C_{\tau\cap\pi}$ is a reducible hyperplane section of S_π, formed by two smooth curves of respective genus 0 and 1 intersecting in 3 points: it follows that the general hyperplane section is a smooth curve of genus $0+1+3-1=3$.
The intersection of S^σ and S_π is a hyperplane section of S^σ, a curve of degree 9. So the two curves obtained cutting S^σ and S_π with a general hyperplane intersect in 9 points.
The sectional genus of M_4^{22}, which is the genus of a hyperplane section of $S^\sigma \cup S_\pi$, is equal to $1+3+9-1=12$.

M_4 is Fano

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Proposition

A general curve section of M_4^{22} is therefore a non-degenerate smooth curve of genus 12 in $\mathbb{P}^{14-3=11}$ of degree 22; by Riemann-Roch this is a canonical curve, i.e. it is embedded by its complete canonical system.

Summarizing we have the following Proposition.

Proposition

The above described 4-fold $M_4 (= M_4^{22}) \subset M_8^{14}$ is an irreducible smooth variety of dimension 4 with canonical sectional curves.
In particular it is a Fano 4-fold of index 2, i.e. $-K_{M_4} = 2H$, where H is the hyperplane bundle of the Grassmannian M_8^{14}.
A very general hyperplane section M_3 of M_4, by Bertini theorem, is a smooth 3-fold whose curve section is canonical.
It is a smooth Fano 3-fold of degree 22 in $\mathbb{P}^{13}$.

Generalized Fano's construction

Let S be any smooth projective variety and consider the Hilbert Scheme which parametrizes its zero dimensional subschemes of length 2, $S^{[2]}$. Let $\varphi : S^{[2]} \to S^{(2)}$ be the Hillb to Chow map; it contracts a divisor $D \subset S^{[2]}$ to a surface in $S^{(2)}$.

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Choose an embedding $S \hookrightarrow \mathbb{P}^N$ and consider the natural map $S^{[2]} \to G(1, N)$ associating to each subscheme of length 2 of S the unique line containing its image in $\mathbb{P}^N$. Compose further with the Plücker embedding of the Grassmannian, $Gr(1, N) \to \mathbb{P}^{\frac{(N+1)N}{2}-1}$. The image is the variety of the lines that are secants or tangents to $S \subset \mathbb{P}^N$.

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By the classical so called trisecant lemma, if $N \geq 4$ the total map $S^{[2]} \to \mathbb{P}^{\frac{(N+1)N}{2}-1}$ is a birational map (onto its image).

The case of $(\mathbb{P}^1 \times \mathbb{P}^1)^{[2]}$

Proposition

Let $\mathcal{H}$ *be the Hilbert Scheme* $(\mathbb{P}^1 \times \mathbb{P}^1)^{[2]}$

a) $\mathcal{H}$ *is a smooth projective variety of dimension* 4

b) $Pic(\mathcal{H}) = \mathbb{Z}(H_1^{[2]}) \oplus \ Z(H_2^{[2]}) \oplus \mathbb{Z}(B/2)$

c) $Nef(\mathcal{H})$ *is the simplicial cone:* $< H_1^{[2]}, H_2^{[2]}, H_1^{[2]} + H_2^{[2]} - (B/2) >$

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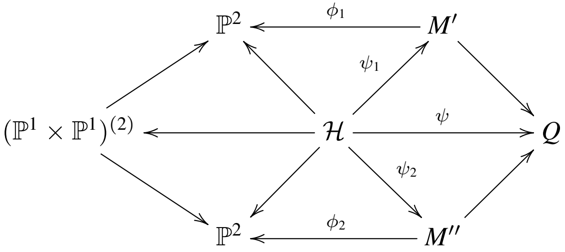
The diagram represents the maps associated to the nef bundles on $\mathcal{H}$.



Take the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$, given by the complete linear system $(1, 1)$, as a smooth quadric surface $Q_2 \subset \mathbb{P}^3$.
Note that that the secant lines fill up the whole Grassmannian $G(1, 3)$, since every line in $\mathbb{P}^3$ is secant to any quadric surface. The Plücker embedding maps $G(1, 3)$ into a (Klein) quadric 4−fold Q_4 in $\mathbb{P}^5$.
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This is the map ψ in the above diagram; it contracts two disjoint divisors, D_1 and D_2, to two conics, $C_1, C_2 \subset Q_4$, which describe in the Grassmannian the lines in the ruling.
All non zero dimensional fibers of ψ are isomorphic to $\mathbb{P}^2$, in particular they all have the same dimension. By a general result, [A-Wisniewski], $\psi : \mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1) \to Q_4 \subset \mathbb{P}^5$ is the blow up of the quadric $Q_4 \subset \mathbb{P}^5$ along two disjoint smooth conics, C_1, C_2.

Fano's Construction revisited

Embed $S = \mathbb{P}^1 \times \mathbb{P}^1$ by the linear system $(1, 2)$ as the normal rational scroll of degree 4, $R^4 \subset \mathbb{P}^5$.
The birational map $\psi_1 : \mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1) \to M'$ **is, by construction, exactly the one in Fano's paper**: $M' = M_4 \subset \mathbb{P}^{14}$.

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In this case the map contracts only one of the two above mentioned divisors, namely the one corresponding to the ruling in lines of R^4, which we denote with D_2; therefore M_4 is smooth.
The other divisor $E := D_1$ remains isomorphically equal in M_4 and it can be contracted as a smooth blow-down to the curve $C_1 \subset Q_4 \subset \mathbb{P}^5$, $\nu \colon M_4 \to Q_4$. C_1 is a smooth conic (not contained in any plane $\mathbb{P}^2 \subset Q_4 \subset \mathbb{P}^5$).

A new description

Let $\nu\colon M_4 \to Q_4$ be the blow-up of a smooth conic $C_1 \subset Q_4 \subset \mathbb{P}^5$ (not contained in any plane $\mathbb{P}^2 \subset Q_4 \subset \mathbb{P}^5$).

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Let H be the hyperplane bundle in $\mathbb{P}^5$; the formula for the canonical bundle of the blow up gives

$$-K_{M_4} = \nu^*(4H) - 2E = 2(\nu^*(2H) - E).$$

The line bundle $\mathcal{L} := \nu^*(2H) - E$ is very ample; it embeds M_4 into $\mathbb{P}^{14}$ as a Fano manifolds of index 2 and genus 12.

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In the classification obtained by Mukai of Fano 4- folds of index 2 (coindex 3 in Mukai' notation) one can find M_4, given as the blow-up of a four dimensional quadric along a conic, as the only one of genus 12 (Example 2). The classification was based on Conjecture (ES) which was later proved by Mella.

Mori-Mukai classification

A general hyperplane section in $\mathcal{L} = \nu^*(2H) - E$ is a Fano $3-$fold, which we denote as Fano did with M_3^{22}. Since, as we have seen above, $\mathcal{L}$ embeds M_4 as the image of Q_4 by the rational map given by the quadric hypersurfaces through a general (=not contained in a plane) conic in Q_4, the hyperplane section M_3^{22} is obtained blowing up the conic in the intersection of Q_4 with another quadric containg the conic.

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This proofs that the M_3^{22}, Fano's last Fano, is the number 16 in the Mori-Mukai list of Fano 3-folds with Picard number 2. In fact they describe this case as the blow up along a conic of a complete intersection of two quadrics in $\mathbb{P}^5$.