The Problem of Malfatti: Two Centuries of Debate

Marco Andreatta, András Bezdek and Jan P. Boroński

Gianfrancesco Malfatti (Figure 1) was a brilliant Italian mathematician born in 1731 in a small village in the Italian Alps, Ala, near Trento. He first studied at a Jesuit school in Verona, then at the University of Bologna. Malfatti was one of the founders of the Department of Mathematics of the University of Ferrara. He died in Ferrara in 1807.

As a very active intellectual in the Age of Enlightenment, he devoted himself to the promotion of many new ideas and wrote many papers in different fields of mathematics including algebra, calculus, geometry, and probability theory. He played an important role in the creation of the *Nuova Enciclopedia Italiana* (1779), in the spirit of the French Encyclopédie edited by Diderot and d'Alembert. His mathematical papers were collected by the Italian Mathematical Society in the volume [7]. His historical figure has been discussed in a series of papers in [1].

This paper was inspired by a conference in 2007 commemorating the 200th anniversary of Malfatti's death, organized by the municipality of Ala and the mathematics departments of Ferrara and Trento.

Malfatti appears in the mathematical literature of the last two centuries mostly in connection with a problem he raised and discussed in a paper in 1803 (Figure 2): how to pack three non-overlapping circles of maximum total area in a given triangle? Malfatti assumed that the solution consisted of three mutually tangent circles, each also tangent to two edges of the triangle (now called Malfatti's arrangement) and in his paper he constructed such arrangements (for a historical overview see [3]). In 1994 Zalgaller and Los [11] disproved Malfatti's original assumption and showed that the greedy arrangement is always the best one. The detailed story of this 200-year-old problem is worth telling because it has many paradigms typical of research in mathematics, including the way one formulates a problem, how one interprets it or solves it, and what one should consider trivial and what one should not.

Figure 1. Gianfrancesco Malfatti (1731–1807).

In the following section we give the history of the problem. The section after that contains a new non-analytic solution for the problem of maximizing the total area of two disjoint circles contained in a given triangle. In the last section we generalize the two-circle problem for certain regions other than triangles. Our non-analytic approach shows that in various situations the greedy arrangements are the best ones.

Malfatti's Marble Problem and Its History

The term *stereotomy* in the title of Malfatti's paper (from the Greek *stereo* = στερεο, which means solid, rigid and

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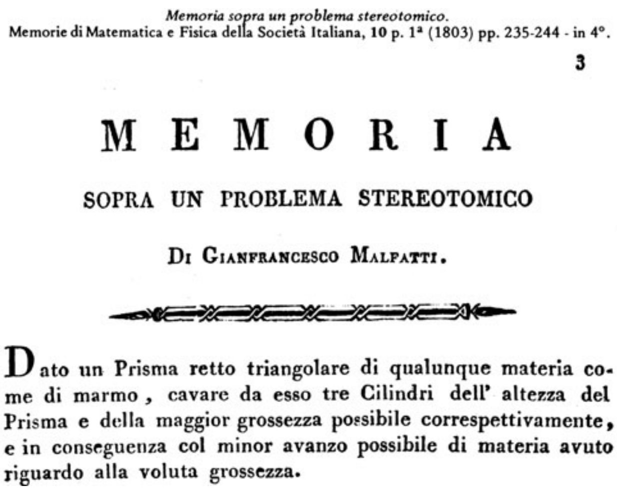

Memoria sopra un problema stereotomico.
Memorie di Matematica e Fisica della Società Italiana, 10 p. 1ª (1803) pp. 235-244 - in 4°.

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M E M O R I A

SOPRA UN PROBLEMA STEREOTOMICO

DI GIANFRANCESCO MALFATTI.

Dato un Prisma retto triangolare di qualunque materia come di marmo, cavare da esso tre Cilindri dell' altezza del Prisma e della maggior grossezza possibile correspettivamente, e in conseguenza col minor avanzo possibile di materia avuto riguardo alla voluta grossezza.

Figure 2. Title page of the paper "On a stereotomy problem".

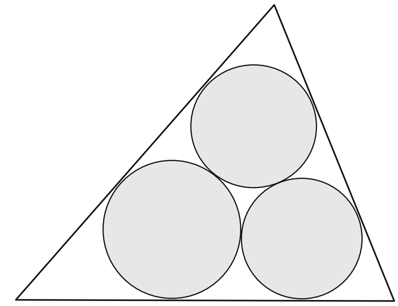


Figure 3. Malfatti's triplet.

tomy = τομ*ia*, which means cut, section) refers to *the art of cutting solids into certain figures or sections, as arches, and the like;* it refers especially to *the art of stonecutting*.

The first lines of the paper are more specific about the problem: "… *given a triangular right prism of whatsoever material, say marble, take out from it three cylinders with the same heights of the prism but of maximum total volume, that is to say with the minimum scrap of material with respect to the volume* …". We summarize the rest of the page with some observations.

1. Malfatti noted that his problem can be reduced, via a stereotomy, to a problem in plane geometry. Though not explicitly stated in the paper, the reduced problem is: *Given a triangle find three non-overlapping circles inside it of total maximum area*. The literature refers to this problem as *Malfatti's marble problem*.
2. Then, without any justification, Malfatti "…*observed that the problem reduces to the inscription of three circles in a triangle in such a way that each circle touches the other two and at the same time two sides of the triangle*…". Today we know that Malfatti's intuition was wrong: this geometric configuration (Figure 3), "Malfatti's configuration," does not solve his marble problem. Yet the remaining part of Malfatti's work was correct.
3. Malfatti constructed the unique three-circle arrangement that today bears his name. In his words "…*Undertaken therefore the solution of this* **second** *problem, I found myself plunged into prolix calculations and harsh formulae*…".

The progress made on Malfatti's marble problem and Malfatti's construction problem should be separated from each other.

Malfatti's construction problem: It is believed that Jacob Bernoulli considered this question for isosceles triangles a century before Malfatti. The problem can also be found in Japanese temple geometry, where it is attributed to Chokuyen Naonobu Ajima (1732–1798). Malfatti's approach was algebraic. He computed the coordinates of the centers of the circles involved, and noticed that the values of the expressions can be constructed using ruler and compass (the reader can find the explicit solution of Malfatti in his paper [6] and also in Section 5 of the more recent book [8]). In 1826 Steiner published an elegant solution of Malfatti's construction problem. He also considered several variations,

AUTHORS

MARCO ANDREATTA is Professor of Geometry at the University of Trento, Italy. He received his "Laurea" in 1981. After post docs in Italy and the United States, he taught at the University of Milan before coming to Trento in 1991. His main field of research is algebraic geometry. He lives in the Italian Alps, very near to Malfatti's birth place; besides mathematics he likes walking, cycling, and skiing in the mountains.

Dipartimento di Matematica
Universitá di Trento
via Sommarive, 38123 Trento
Italy
e-mail: marco.andreatta@unitn.it

ANDRÁS BEZDEK is Professor of the Mathematics at Auburn University, Alabama, and Senior Research Fellow at the Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, Hungary. He received his PhD degree from Ohio State University in Columbus in 1986 and his Doctor of Academy title (DSc) from the Hungarian Academy of Sciences in 2006 for his work in discrete geometry. He enjoys swimming and playing tennis.

MTA Rényi Institute
13-15 Réaltanoda u.,
Budapest
Hungary
e-mail: bezdean@auburn.edu

including analogous problems where the sides of the triangle are replaced by circular arcs, or when these arcs are placed on a sphere. In 1811 Gergonne asked about the existence of a similar extremal arrangement in three-dimensional space, using a tetrahedron and four spheres instead of a triangle and three circles. The extremal arrangement of spheres was constructed by Sansone in 1968. In the nineteenth century many mathematicians, including Cayley, Schellbach, and Clebsch, worked on various generalizations.

Malfatti's marble problem: In 1930 Lob and Richmond [10] observed that in an equilateral triangle the triangle's inscribed circle together with two smaller circles, each inscribed in one of the three components left uncovered by the first circle, produces greater total area than Malfatti's arrangement. Eves [2] pointed out that in a very tall triangle placing three circles on top of each other also produces greater total area. We say that n circles in a given region form a *greedy arrangement*, if they are the result of the n-step process, where at each step one chooses the largest circle which does not overlap the previously selected circles and is contained by the given region. Goldberg [5], see also [4], outlined a numerical argument that the greedy arrangement is always better than Malfatti's. Mellisen conjectured, in [9].

Conjecture 1 *The greedy arrangement has the largest total area among arrangements of n non-overlapping circles in a triangle.*

Malfatti's marble problem is the case $n = 3$; it was settled by Zalgaller and Los [11].

Following [11] we say that a system of n non-overlapping circles in a triangle is a *rigid arrangement* if it is not possible to continuously deform one of the circles in order to increase its radius, without moving the others and keeping all circles non-overlapping.

JAN P. BOROŃSKI graduated from Silesian University at Katowice, Poland, majoring in Mathematics in 2005, and he received his PhD from Auburn University, Alabama, in 2010, under the guidance of Professor Krystyna Kuperberg. At Auburn he was a recipient of College of Sciences and Mathematics Dean's Award for Outstanding PhD Graduate Student Research. His research interests include topological fixed-point theory, dynamical systems, and continuum theory. Among his hobbies are mountain biking, skiing, hiking, music, and movies.

Mathematics & Statistics
Auburn University
Auburn, AL 36849-5310
USA
e-mail: boronjp@auburn.edu

It is evident that the solution of Malfatti's marble problem is in the class of rigid arrangements. Note also that every circle of a rigid arrangement has at least three points of contact either with the other circles or with the sides of the triangle; moreover, these points of contact do not lie on one closed semicircle of the boundary of the circle under consideration. Zalgaller and Los showed, by an elaborate case analysis, that if $n = 3$, then with the exception of the greedy triplet, all rigid configurations allow local area improvements.

Two Circles in a Triangle

We now consider the Malfatti marble problem for $n = 2$, arranging two non-overlapping circles of maximum total area in a given triangle.

Theorem 1 *The greedy arrangement has the largest total area among pairs of non-overlapping circles in a triangle.*

This problem is not difficult; an analytic solution is explained in [9] and a similar solution is also included implicitly in the work of Los and Zalgaller [11]. We present a new non-analytic solution, which will lead to several generalizations. Let us note first that the greedy arrangement consists of the inscribed circle and the one touching the two longer sides and the inscribed circle.

Proof of Theorem 1 Let ABC be the given triangle. Assume that two tangent circles are arranged in the triangle ABC so that the first circle touches the sides AB and AC and the second circle touches the sides AC and BC (Figure 4). This is the rigid arrangement of two circles. Let r be the radius of the first and let R be the radius of the second circle. If both circles are held fixed by their contact points, then R is uniquely determined by r. Denote by $R(r)$ the function that describes the relation between the radii r and R.

We will prove that the total area function $(r^2 + R^2(r))\pi$ is convex. Therefore the area function attains its maximum at the end points of the admissible interval of r, which implies that the greedy arrangement is the best.

A real-valued function $f(x)$ is *midpoint-convex* on an interval if for any two numbers x, x' from its domain, $f\left(\frac{x+x'}{2}\right) \le \frac{f(x)+f(x')}{2}$. It is known that any continuous,

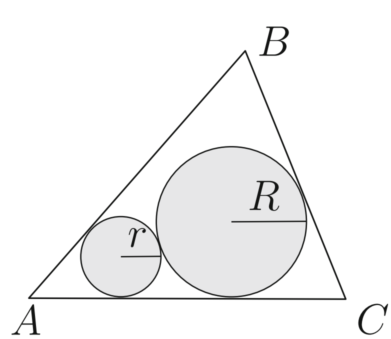


Figure 4. Two circles' rigid arrangement.

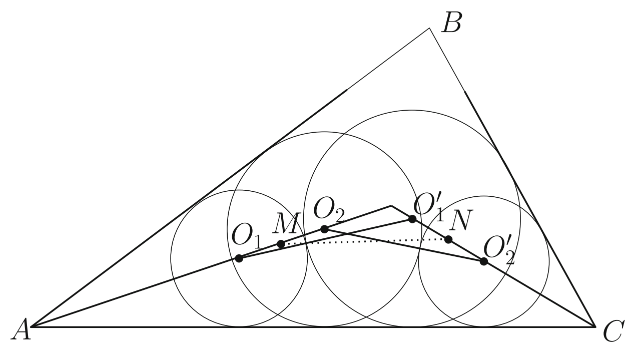


Figure 5. Comparison of two circles' rigid arrangements.

midpoint-convex function is convex. It is also known that if both $f(x)$ and $g(x)$ are convex functions, then i) $f(x) + g(x)$ is also convex, furthermore ii) if in addition to being convex, $f(x)$ is also increasing, then $f(g(x))$ is convex. Thus all we need to show is:

Lemma 1 *$R(r)$ is a midpoint-convex function of r.*

Let r_1, $R(r_1)$ and r_2, $R(r_2)$ be the radii of two pairs of circles satisfying the conditions of Lemma 1. Denote by O_1, O_1' and similarly by O_2, O_2' the centers of these circles (see Figure 5).

Clearly the following equalities hold:

$$|O_1O_1'| = r_1 + R(r_1) \quad \text{and} \quad |O_2O_2'| = r_2 + R(r_2)$$

Let us recall the following elementary geometric exercise,

Exercise 1 *Show that in any quadrilateral the sum of the lengths of two opposite sides is at least twice the distance between the midpoints of the remaining two sides.*

Solution of Exercise 1: Let $ABCD$ be any quadrilateral (Figure 6); it can be convex, concave, or self-intersecting and can have collinear or even coinciding vertices. Let M and N be the midpoints of side AB and CD. Reflect B through the midpoint N to get B'. Obviously $2|MN| = |AB'| \leq |AD| + |DB'| = |AD| + |BC|$, which is what we wanted to show.

Applying this exercise to the quadrilateral $O_1O_2O_2'O_1'$, with $O_1O'_1$, and O_2O_2' being the opposite sides and M and N being the midpoints of the two remaining sides, we get

$$|MN| < \frac{|O_1O_1'| + |O_2O_2'|}{2} = \frac{r_1 + r_2}{2} + \frac{R(r_1) + R(r_2)}{2}.$$

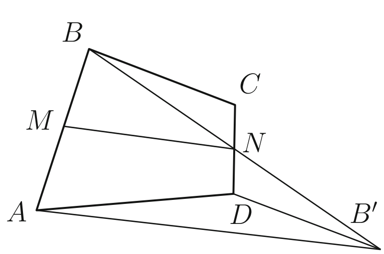


Figure 6. Exercise 1.

In other words, the circle centered at M of radius $\frac{r_1+r_2}{2}$ and the one centered at N of radius $\frac{R(r_1)+R(r_2)}{2}$ must overlap. Consequently,

$$R\left(\frac{r_1 + r_2}{2}\right) < \frac{R(r_1) + R(r_2)}{2}.$$

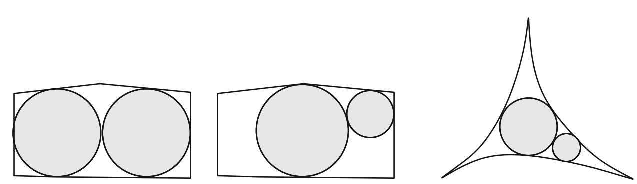
Two Circles in Other Regions and Other Generalizations

It is natural to ask whether the greedy arrangement still gives the largest total area when the circles are placed in regions other than a triangle. Mellisen [9] showed a pentagon (Figure 7a) where the greedy arrangement clearly does not win. In this section we call a region a *concave triangle* if it is bounded by three concave curves (Figure 7b). We will prove

Theorem 2 *The greedy arrangement has the largest total area among pairs of non-overlapping circles in any concave triangle* (Figure 7b).

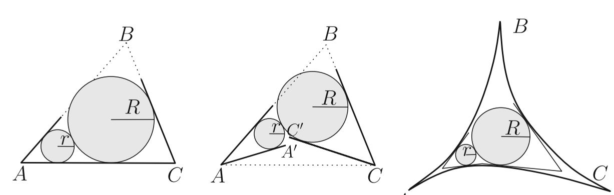
When we proved Theorem 1 we looked at a pair of (rigid) circles which were different from the greedy arrangement and which could not be improved by changing only one of them (see Figure 4 again). We considered a continuous change of the two circles. It turned out (Lemma 1) that as we changed the radius r of one of the circles, the radius $R(r)$ of the other circle, as a function, changed in a convex manner. This essentially meant that the total area could be improved locally. The bottom line is that any generalization of Lemma 1 could lead to a new theorem.

First of all, the proof of Lemma 1 remains true word by word if the two circles are not restricted to the triangle (Figure 8a). The exact same proof remains valid if the two circles are allowed to increase or decrease maintaining contact not with the sides of the triangle but with the sides of two angular sectors (Figure 8b). Formally we have



(a) Melissen's pentagon **(b)** Concave triangle

Figure 7. Convex and concave containers.



(a) Modified Lemma 1 **(b)** Lemma 2 **(c)** Theorem 2

Figure 8. Proof of Theorem 2.

Lemma 2 *Let AA' and CC' be two non-intersecting segments in a triangle ABC. Assume two tangent circles of radii r, R are given so that the first circle touches the sides AB and the interior of the segment AA', and the second circle touches the sides BC and the interior of the segment CC'. If both circles are held fixed by the contact points, then R is determined by r and the function which describes this relation is convex in r.*

Proof of Theorem 2, by contradiction. Assume that the maximum total area is achieved by a pair of circles different from the greedy arrangement. Then the circles touch each other and they also touch exactly two of the concave curves (Figure 8c). The tangent lines of the circles at these contact points (if they are not unique, choose any of them) together with the circles satisfy the conditions of Lemma 2 and thus allow local area improvement.

Assume one needs to arrange two non-overlapping spheres of greatest total volume in a given tetrahedron. The centers of spheres which are touching the same three faces of the tetrahedron are on a line and lines corresponding to different triples of the faces meet at the incenter of the tetrahedron. Thus the two-sphere marble problem leads to Lemma 1 again, and we have

Remark 1 *The greedy arrangement has the largest total area among pairs of non-overlapping spheres in a tetrahedron.*

Assume one needs to arrange two non-overlapping circles of greatest total area in a triangle of the hyperbolic plane. Among steps of the Euclidean proof of Theorems 1 and 2 only the elementary geometric fact "for any quadrilateral (which can be convex, concave or self intersecting, or degenerate) the sum of the lengths of two opposite sides is at least twice the distance between the midpoints of the remaining two sides" needs to be questioned. Since that inequality holds in the hyperbolic plane too, we have

Remark 2 *Theorem 1 and 2 remain true in the hyperbolic plane.*

Consider the analogous problem of placing two circles in a spherical triangle. A straightforward computation shows that in a lune with sufficiently small angle the symmetrical arrangement has larger total area than that of the the greedy arrangement (Figure 9). The reason that the standard proof fails now is the above-mentioned elementary geometric inequality, which on the sphere can be proved only for self-intersecting quadrilaterals. This, in view of the application, means (details are omitted here) that

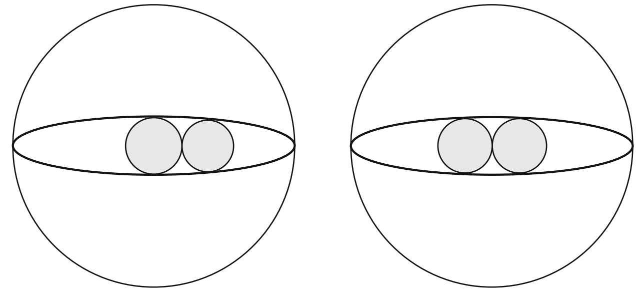


Figure 9. Two circles in a spherical line.

Remark 3 *Theorem 1 remains true on the sphere if the diameter of the spherical triangle is less than $\frac{\pi}{4}$.*

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