The problem of Malfatti two centuries of debate

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From Italy to Russia

Gianfrancesco Malfatti
Ala and Ferrara
Italy (1731 - 1807)

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Viktor Zalgaller
Parfino and Saint Petersburg
Russia (1920-)

Gianfrancesco Malfatti ...

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He played an important role in the creation of the Nuova Enciclopedia Italiana (1779), in the spirit of the French Encyclopédie edited by Diderot and d'Alambert.

.. and his problem

Memoria sopra un problema stereotomico.
Memorie di Matematica e Fisica della Società Italiana, **10** p. 1ª (1803) pp. 235-244 - in 4°.

MEMORIA

SOPRA UN PROBLEMA STEREOTOMICO

DI GIANFRANCESCO MALFATTI.

Dato un Prisma retto triangolare di qualunque materia come di marmo, cavare da esso tre Cilindri dell' altezza del Prisma e della maggior grossezza possibile correspettivamente, e in conseguenza col minor avanzo possibile di materia avuto riguardo alla voluta grossezza.

.. and his problem

The title contains the word: *Stereotomy*, from the Greek
stereo $= \sigma\tau\epsilon\rho\epsilon o$, which means solid, rigid and
tomy= $\tau o \mu ia$, which means cut, section.

Refers to *the art of cutting solids into certain figures or sections, as arches, and the like; the art of stonecutting.*

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“. . . given a triangular right prism of whatsoever material, say marble, take out from it three cylinders with the same heights of the prism but of maximum total volume, that is to say with the minimum scrap of material with respect to the volume. . . ”

Malfatti solution

Malfatti noted that his problem can be reduced, via a stereotomy, to a problem in plane geometry. Though not explicitly stated in the paper, the reduced problem is:
Given a triangle find three non overlapping circles inside it of total maximum area.
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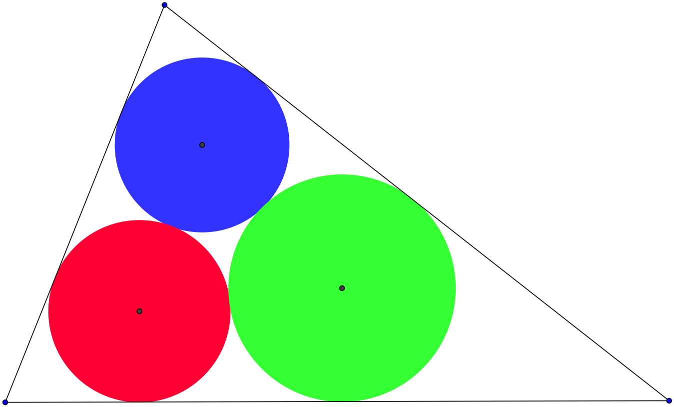
This geometric configuration is called the *Malfatti's configuration*.

Malfatti's configuration



Inscription of three circles in a triangle in such a way that each circle touches the other two and at the same time two sides of the triangle

Construction of the Malfatti's configuration

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Many famous mathematicians worked on this construction and its generalization, including Steiner, Cayley, Schellbach, and Clebsch.
In 1811 Gergonne asked about the existence of a similar extremal arrangement in three dimensional space, using a tetrahedron and four spheres instead of a triangle and three circles. The extremal arrangement of spheres was constructed by Sansone in 1968.

The Malfatti's configuration does not solve the marble problem!

In 1930 Lob and Richmond observed that in an equilateral triangle the triangle's inscribed circle together with two smaller circles, each inscribed in one of the three components left uncovered by the first circle, produces greater total area than Malfatti's arrangement !

The Greedy Arrangement

We say that three (or generically n) circles in a given region form a *greedy arrangement*, if they are the result of the 3-step (n-step) process, where at each step one chooses the largest circle which does not overlap the previously selected circles and is contained by the given region.

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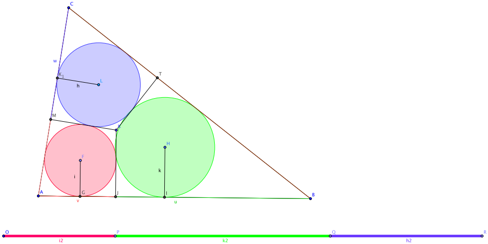
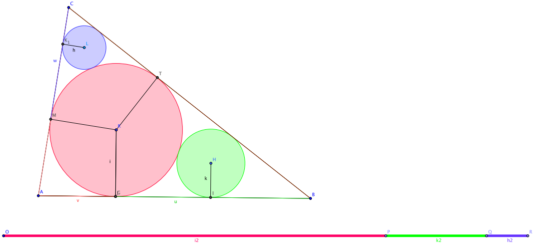
In 1994 Los and Zalgaller settled the Malfatti's marble problem showing that the greedy arrangement is better then any other possible configuration of three non overlapping circles.

Malfatti's versus Greedy's





Open Problems and a Conjecture

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In can be conjectured:

Conjecture. The greedy arrangement has the largest total area among arrangements of n non-overlapping circles in a triangle.

Greedy Algorithm

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Also to find the shortest paths from a single source vertex to any other vertices in a weighted, directed graph, as our car navigator does.

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If there is no greedy algorithm that always finds the optimal solution for a problem, one may have to search (exponentially) many possible solutions to find the optimum. Greedy algorithms are usually quicker.

Some observations

As in Loss and Zalgaller we say that a system of n non-overlapping circles in a triangle is a *rigid arrangement* if it is not possible to continuously deform one of the circles in order to increase its radius, without moving the others and keeping all circles non-overlapping.
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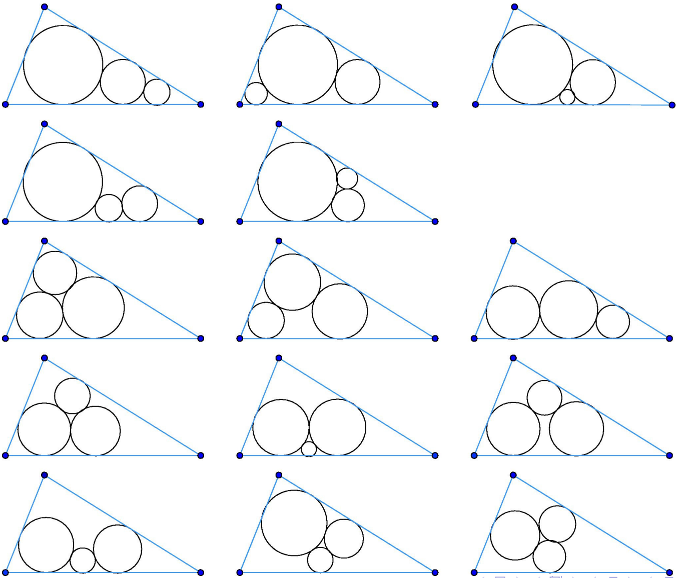
Zalgaller and Los showed, by an elaborate case analysis, that if $n = 3$, then with the exception of the greedy triplet, all rigid configurations allow local area improvements.

Rigid Arrangements of 3-circles



Malfatti's for two circles

Theorem: the greedy arrangement has the largest total area among pairs of non-overlapping circles in a triangle.

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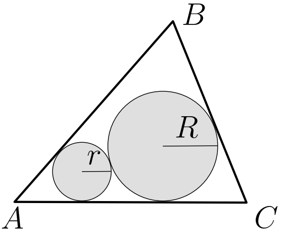


The figure represents a rigid arrangement of two circles. Let r, R be the two radius; then R is uniquely determined by r, that is $R = R(r)$.
We will prove that the total area function $(r^2 + R^2(r))\pi$ is convex.
Therefore the area function attains its maximum at the end points of the admissible interval of r, i.e. the greedy arrangement is the best.

Malfatti's for two circles

A real valued function $f(x)$ is *midpoint convex* on an interval if for any two numbers x, x' from its domain, $f\left(\frac{x+x'}{2}\right) \leq \frac{f(x)+f(x')}{2}$.

Facts:

i)Any continuous, midpoint convex function is convex.

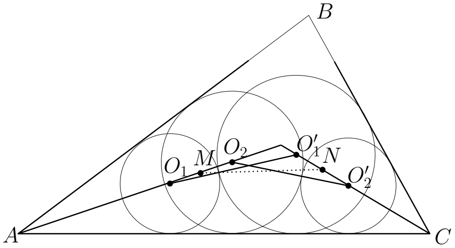
ii) If both $f(x)$ and $g(x)$ are convex functions, then $f(x)+g(x)$ is also convex.

iii) if in addition to convex $f(x)$ is also increasing, then $f(g(x))$ is convex.

Thus all we need to show is:
$R(r)$ is a midpoint convex function of r.

Malfatti's for two circles

Let $r_1, R(r_1)$ and $r_2, R(r_2)$ be the radii of two pairs of circles satisfying a rigid arrangement. Denote by O_1, O_1' and similarly by O_2, O_2' the centers of these circles.



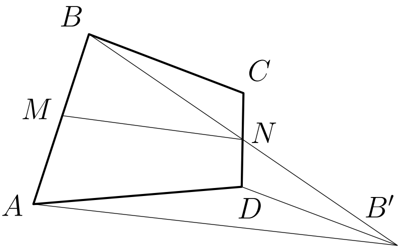
Clearly the following equalities hold:

$$|O_1O_1'| = r_1 + R(r_1) \quad \text{and} \quad |O_2O_2'| = r_2 + R(r_2)$$

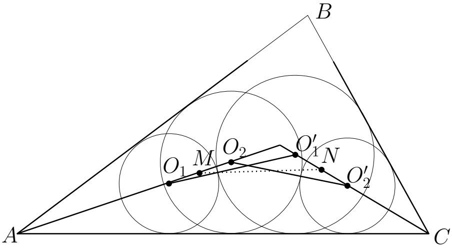
Malfatti's for two circles

Let us recall the following elementary geometric exercise:
in any quadrilateral the sum of the lengths of two opposite sides is at least twice the distance between the midpoints of the remaining two sides.
This follows from the following picture noticing that

$$2|MN| = |AB'| \leq |AD| + |DB'| = |AD| + |BC|$$



Malfatti's for two circles



Consider the quadrilateral $O_1O_2O'_2O'_1$; $O_1O'_1$ and $O_2O'_2$ being the opposite sides, M and N being the midpoints of the two remaining sides:

$$|MN| < \frac{|O_1O'_1| + |O_2O'_2|}{2} = \frac{r_1 + r_2}{2} + \frac{R(r_1) + R(r_2)}{2}.$$

In other words the circle centered at M of radius $\frac{r_1+r_2}{2}$ and the one centered at N of radius $\frac{R(r_1)+R(r_2)}{2}$ must overlap, that is

$$R\left(\frac{r_1 + r_2}{2}\right) < \frac{R(r_1) + R(r_2)}{2}. \qquad \square$$

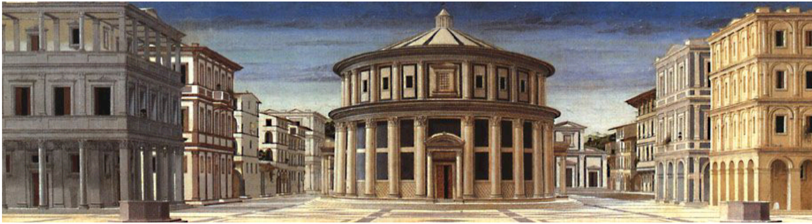
Italian Painters

The mathematical discipline called Projective Algebraic Geometry starts with the work of the italian painters which invented the Projective Space

1483 - Paolo Uccello

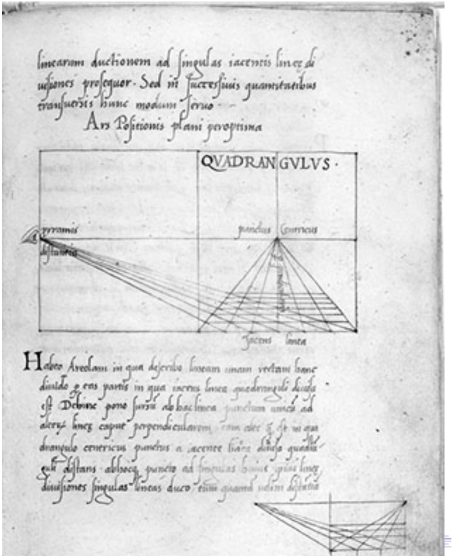
Italian Painters

1483 - Piero della Francesca



Italian Painters

1435 - Leon Battista Alberti



Fano Varieties

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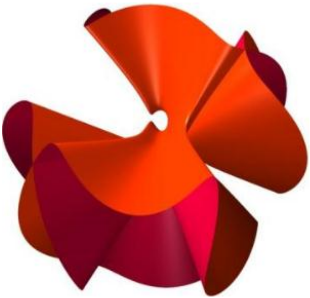
This will be extremely useful for mathematicians, physicists, biologists and other scientists to explore a range of areas, including computer vision, number theory, theoretical physics and synthetic biology.

Some ”stereotomies” of Fano varieties

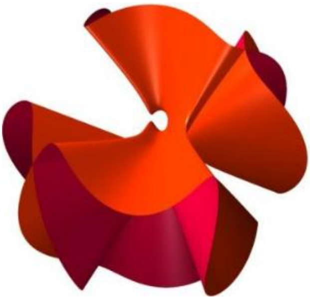


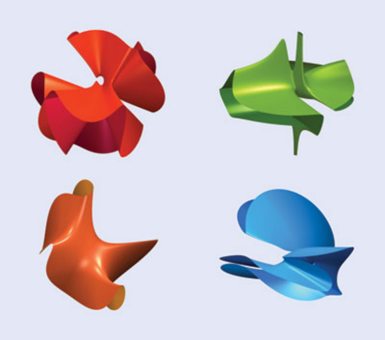
Fano Varieties

Math for All Science

Marco Andreatta

Some ”stereotomies” of Fano varieties





Enriques and Fano

Projective Algebraic Geometry of surfaces and of higher dimension varieties, the old italian school:

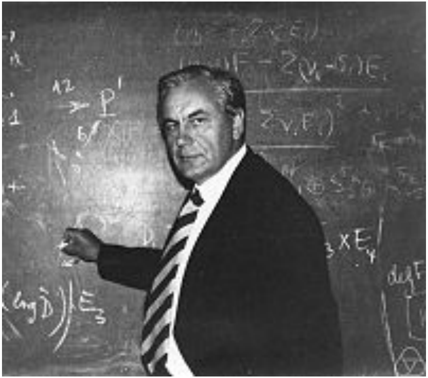
1871-1946 Federigo Enriques

1871-1952 Gino Fano

Shafarevich and Iskowski

The russian school:

Igor R. Shafarevich 1923-



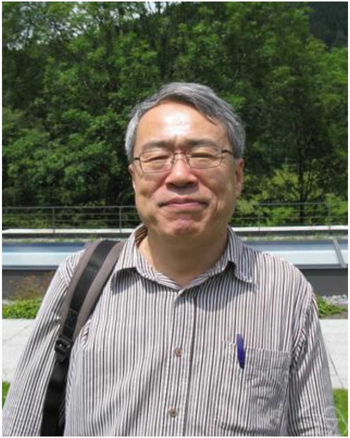
Vasilii A. Iskovskikh1939-2009

Mori and Mukai

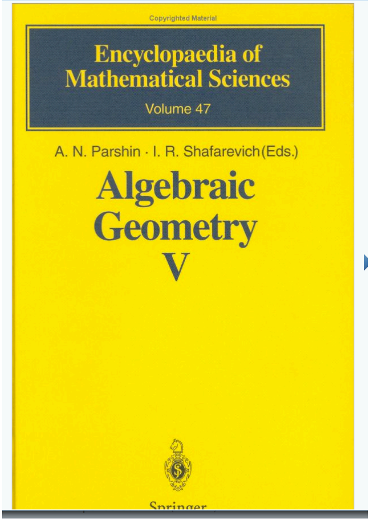
The japanese school:

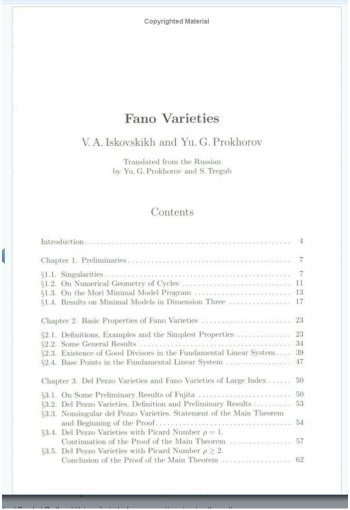
S. Mori 1951-

S. Mukai1953-



Encyclopaedia





Fano Varieties

V. A. Iskovskikh and Yu. G. Prokhorov

Translated from the Russian
by Yu. G. Prokhorov and S. Tregub

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