Gino Fano and modern Projective Geometry

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Beijing-2025

Introduction

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Algebraic Curves- Riemann Surfaces

Theorem (Riemann-Poincaré). A projective variety X with $dim_{\mathbb{C}} X = 1$ is isomorphic to one of the following

- $\mathbb{P}^1_{\mathbb{C}} = S^2$, *the Gauss sphere*;
  TX admits an hermitian metric with positive constant curvature
- $\mathbb{C}/\Gamma$, $\Gamma = \{a + \tau b\}$ with $Im\tau > 0$, *Elliptic Curve*;
  TX admits an hermitian metric with zero curvature.
- $\mathbb{D}/\pi_1(X)$, $\mathbb{D}$ is the *Hyperbolic Disc, by Beltrami-Klein-Poincaré*.
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Theorem (Andreatta-Wisniewski 2001; Campana -Peternell conjecture).
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Fano's Varieties

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Varietà algebriche a tre dimensioni a curve sezioni canoniche,
i.e. projective 3-fold $X \subset \mathbb{P}^N$ such that for general hyperplanes H_1, H_2 the curve $\Gamma := X \cap H_1 \cap H_2$ is canonically embedded.

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If and only if the linear system, $|-K_X]$, embeds X as a 3-fold of degree $2g-2$ into a projective space of dimension $g+1$, where $g = g(\Gamma)$:

$$X := \mathbf{X_3^{2g-2}} \subset \mathbb{P}^{\mathbf{g+1}}$$

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More generally Fano considered $X \subset \mathbb{P}^N$ projective n-fold such that for general hyperplanes $H_1, H_2, ..., H_{n-1} \in |H|$, the curve

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If and only if

$$-K_X = (n-2)H$$

(Fano variety of index $(n-2)$).

Fano's 3folds

$n = 1$: $\mathbb{P}^1$;
$n = 2$: del Pezzo Surfaces (blow-ups of $\mathbb{P}^2$ and the quadric);
$n = 3$, classified by Fano, Iskovskihk, Shokurov, Mukai, Mori;
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S. Mukai classified all Fano variety of index $(n - 2)$, under the assumption of existence of a good section of $|H|$.
The assumption was then proved by M. Mella.

Some key properties of Fano's Varieties

- There are Fano 3-folds which are non rational (Clemens-Griffiths, Iskovskihk-Manin, Artin-Mumford)

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- Fano Manifolds are the only one which could not admit a Kähler-Einstein metric (Matsushima, Futaki, Tian). They have it if and only if they are K-stable (Donaldson et oth. (2015)).

Opposite to Fano's

A projective variety X, smooth or with mild singularities (terminal or klt) which allows to define mK_X as a line bundle, such that $K_X \cdot C \geq 0$ for all curves $C \subset X$ (that is K_X is *nef*), is a *minimal model*.

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Birkar-Cascini-Hacon-McKernan's Theorem is the best result at the moment (it implies that if K_X is big then X is birational to a MM).

Mori Theory

Summarizing Mori Theory and MMP we say that:
if we have a curve, $C \subset X$, such that $K_X \cdot C < 0$ then the part of the cone of effective curve with $K_X < 0$ is not empty, it is polyhedral with extremal rays generated by rational curves, $R = \mathbb{R}^+[C]$ (Cone Theorem).

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Moreover there exists a surjective morphism $f_R : X \to Y$ into a projective normal variety, with connected fibers and which contracts all curve in R (Contraction Theorem (Mori-Kawamata-Shokurov-...)).
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MMP proceeds inductively with two steps

- taking $f_R(X)$ instead of X if f_R is birational and divisorial
- flipping f_R into $f' : X' \to Y$ is f_R is birational and small

until we reach either a MM or a fiber type contraction $f_R : X \to Y$.

Of course we have to prove existence and termination of flips.

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Kawamata (small (1989) and Andreatta-Wisniewski (general, 1998). If $dimX = 4$ the list is longer, we have a unique small contraction and, besides the obvious blow-ups, some divisorial to surface contractions with jumping fibers.

Classification of birational FM contraction, 3-fold singular case

Mori (1991) and Kollar-Mori (1992) classified all FM small contractions $f_R : X \to Y$ and their flips on a 3-dimensional variety with terminal singularities (starting from Paolo Francia example of 1985).

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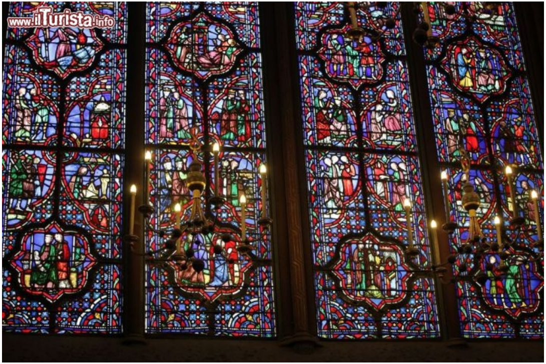

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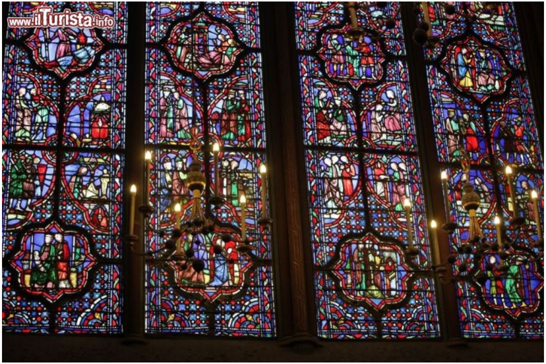

Together with the termination of flips (Shokurov) this gives MMP in dimension 3.

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The classification of FM fiber type contractions (included Fano's variety) on terminal variety of dimension 3 is a long lasting job which is carried on by many groups all over the world.

C. Birkhar recent results states that under some additional assumptions they form a set of bounded families (in all dimensions),.

Classification of birational FM contraction, n-fold case via Adjunction Theory

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Sommese and his school, in order to classify the pairs (X, H), followed a method indicated by Fano and by his student/collaborator Morin, which consists in studying the nefness of $K_X + rH$, with $r > 0$.

In Mori theory language a nef divisor of the type $K_X + rH$ has a FM contraction associated to it, which can be studied.

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This is done *lifting* the classification from the 3-dimensional case; in particular lifting the structures of weighted projective blow-ups, as in the first paper of S. Mori.

Prizes

- S. Mori: Fields Medalist in 1990 for *the proof of Hartshorne's conjecture and his work on the classification of three-dimensional algebraic varieties*
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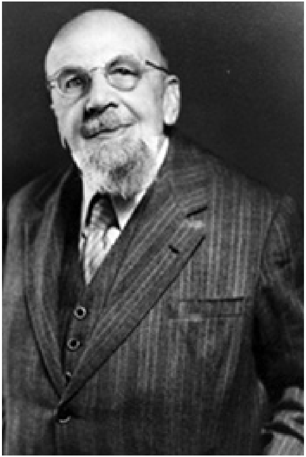

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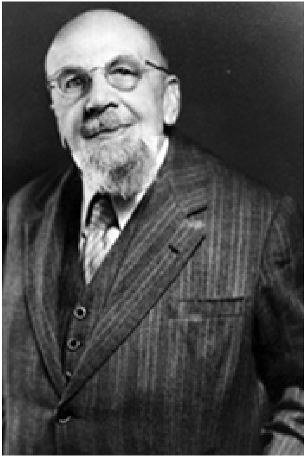

Gino Fano 1871-1952

Geometria algebrica. — *Su una particolare varietà a tre dimensioni a curve-sezioni canoniche.* Nota (*) del Socio GINO FANO.

1. Ho incontrato recentemente una varietà a tre dimensioni a curve-sezioni canoniche, che naturalmente appartiene alla serie delle M_3^{2p-2} di S_{p+1} (qui $p = 12$), oggetto di mie ricerche in quest'ultimo periodo (1), ma non ha finora richiamata particolare attenzione. Ne darò qui un breve cenno.

Consideriamo nello spazio S_5 una rigata razionale normale R^4 (non cono), che per semplicità supponiamo del tipo più generale, cioè con ∞^1 coniche direttrici irriducibili; e con essa la varietà ∞^4 delle sue corde. Quale ne è l'immagine M_4 nella Grassmanniana M_8^{14} di S_{14} (2) delle rette di S_5 (3)?

Determiniamo anzitutto l'ordine di questa M_4, ad esempio l'ordine della superficie sua intersezione con un S_{12}, vale a dire della ∞^2 di rette comune alla ∞^4 suddetta e a due complessi lineari. Valendoci di due complessi costituiti risp. dalle rette incidenti a due S_3, questi ultimi contenuti in un $S_4 \equiv \sigma$ e aventi perciò a comune un piano π, la ∞^2 di rette in parola si spezzerà nei due sistemi delle corde di R^4 contenute in σ e di quelle incidenti al piano π. Le prime sono le ∞^2 corde di una C^4 razionale normale, e nella Grassmanniana delle rette di σ hanno per immagine una superficie φ^9 di S_9 di Del Pezzo (4). Della seconda ∞^2 prendiamo l'intersezione con un ulteriore complesso lineare, anche con un $S_3 \equiv \tau$ direttore incontrante π in una retta. Si ha una rigata composta di una parte luogo delle corde di R^4 contenute nello spazio $S_4 \equiv \tau\pi$ e incidenti a π, la cui imma-

(*) Presentata nella seduta dell'8 gennaio 1949.

(1) Più specialmente nella Memoria: *Sulle varietà algebriche a tre dimensioni a curve-sezioni canoniche.* « Mem. Acc. d'Italia », classe sc. fis., vol. VIII (1937), n. 2.

(2) F. SEVERI, *Sulla varietà che rappresenta gli spazi subordinati*... « Ann. di Matem. » (3) vol. 24 p. 89 (1915).

(3) Questione analoga per le rigate razionali normali di ordine inferiore: le corde di una quadrica di S_3 costituiscono la totalità delle rette di questo spazio, cioè una M_4^2 di S_5. Il sistema delle corde di una R^3 di S_4 è una M_4^9 di S_9 a curve-sezioni di genere 3, la cui superficie-sezione generica è rappresentata sul piano da un sistema lineare di quartiche con 7 punti base.

(4) Questa ∞^2 contiene infatti una rete omaloidica di rigate cubiche, generate dalle involuzioni di 2° ordine sulla C^4, e tutte irriducibili.

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Fano considered the **smooth rational quartic normal scroll** embedded in $\mathbb{P}^5$ with degree 4,

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Then he defined
M_4 as the subset of the Grassmannian of lines in $\mathbb{P}^5$ given by the *chords* of $S_{2,2}$, embedded via the Plücker embedding in $\mathbb{P}^{14}$.

(The word "chords" is interpreted as secant and tangent lines)

...to prove...

Fano proved the following:

- $M_4 \subset \mathbb{P}^{14}$ is an irreducible smooth variety of dimension 4
- $\deg M_4 = 22$
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Subsequently he took a general hyperplane section of M_4 obtaining a **smooth** 3**-fold,** $M_3 \subset \mathbb{P}^{13}$**, of degree** 22 **and sectional genus** 12.

Modern and general set up; Hilbert Scheme

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Let $\mathbb{F}_n[2]$ to be the Hilbert scheme of length two 0–dimensional subschemes of $\mathbb{F}_n$.
It is a smooth variety of dimension 4 and Picard rank 3 (Fogarty 1973).

Modern and general set up; Hilbert Scheme

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F a fiber, E the section at infinity, $E^2 = -n$.

Let $\mathbb{F}_n[2]$ to be the Hilbert scheme of length two 0–dimensional subschemes of $\mathbb{F}_n$.
It is a smooth variety of dimension 4 and Picard rank 3 (Fogarty 1973).

The Nef Cone of $\mathbb{F}_n[2]$ has three faces associate to elementary contractions (Bertram-Coskun 2013).:

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- The contraction of the the divisor $\mathcal{F}_n$ of all pairs of points on a fiber of π to a smooth, rational curve $\Gamma \subset Z_{n,1}$: $\phi_{n,1} : \mathbb{F}_n[2] \longrightarrow Z_{n,1}$.
$Z_{n,1}$ is smooth and $\phi_{n,1}$ is the blow–up of Γ (Andreatta -Occhetta 2002).

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- The contraction of the surface $\mathcal{E}_n \cong \mathbb{P}^2$ of all pairs of points on E, to a singular point $\mathfrak{p}$.$\phi_{n,2} : \mathbb{F}_n[2] \longrightarrow Z_{n,2}$, for $n \geq 1$,

Projective realization

For a *projective realization* we consider the **smooth rational normal scrolls** $S_{a,b}$ of degree $r = a + b$ in $\mathbb{P}^{r+1}$:
$S_{a,b} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)) \cong \mathbb{F}_{b-a}$.

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Take the morphism

$$\gamma_{a,b} : S_{a,b}[2] \cong \mathbb{F}_{b-a}[2] \longrightarrow \mathbb{G}(1, r+1) \subset \mathbb{P}^{\frac{r(r+3)}{2}}$$

to the Grassmannian of lines in $\mathbb{P}^{r+1}$ in its Plücker embedding:
each 0–dimensional length 2 scheme $\mathfrak{y} \in S_{a,b}[2]$ is mapped by $\gamma_{a,b}$ to the line $\ell_{\mathfrak{y}} := \langle \mathfrak{y} \rangle$ spanned in $\mathbb{P}^{r+1}$ by $\mathfrak{y}$.

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We call $\gamma_{a,b}$ the **secant map of $S_{a,b}$ and its image $X_{a,b}$**.

Fano Last Fano 4-fold

$$
\begin{array}{ccc}
\mathbb{F}_0[2] & \xrightarrow{\gamma_{2,2}} & X_{2,2} \\
\Big\downarrow{\scriptstyle \gamma_{2,2}} & \searrow^{\gamma_{1,1}} & \Big\downarrow{\scriptstyle \psi_{0,1}} \\
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$X_{1,1}$ is $\mathbb{G}(1,3)$, that is a smooth quadric $Q_4 \subset \mathbb{P}^5$.
The morphism $\gamma_{1,1}$ is the blow up along the two conics Γ and Γ' that correspond to the two rulings of $S_{1,1}$.

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$X_{2,2}$ is the Fano M_4 and $\psi_{0,1} : X_{2,2} \to X_{1,1} = Q_4$ is the blow-up of one smooth conic $\Gamma \subset Q_4 \subset \mathbb{P}^5$ not contained in any plane $\mathbb{P}^2 \subset Q_4 \subset \mathbb{P}^5$.

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Let H be the hyperplane bundle in $\mathbb{P}^5$. The line bundle $\mathcal{L} := \nu^*(2H) - E$ is very ample; it embeds $X_{2,2}$ into $\mathbb{P}^{14}$ as a Fano manifolds of index 2 and genus 12. Moreover $-K_{X_{2,2}} = 2\mathcal{L}$.
This is Example 2 in the classification of Mukai of Fano 4- folds of index 2.

The degree of the secant variety $X_{a,b}$

Andreatta-Ciliberto-Pignatelli:
we proved that the degree of $X_{a,b}$ in $\mathbb{P}^{\frac{r(r+3)}{2}}$ $(r = a + b)$ is

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$$\deg(X_{a,b}) = 3r^2 - 8r + 6$$

The formula for $r = 4$ agrees with the Fano computation (22);
we actually computed the degree following Fano's ingenious argument.
(The formula was obtained also by Cattaneo (2020) in other cases)

The singularity of the secant variety $X_{a,b}$

$X_{a,b}$ is smooth, except for $X_{1,r-1}$, in which case it has a unique singular point at $\mathfrak{p}$, whose tangent cone is the cone over a smooth threefold linear section of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^{r-2}$ in $\mathbb{P}^{3r-4}$.

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$X_{1,3}$ can be obtained as follows:
let $\nu \colon \tilde{X} \to X_{1,1} = Q_4$ be the blow-up of a smooth conic $C' \subset Q_4 \subset \mathbb{P}^5$ contained in a plane $\Gamma = \mathbb{P}^2 \subset Q_4 \subset \mathbb{P}^5$.
The contraction of the strict transform of the plane Γ in $\tilde{X}$ gives $\tilde{X} \to X_{1,3}$

FlF in Mori-Mukai classification

Fano last Fano (FlF), according to Fano, is a general hyperplane section in $\mathcal{L} = \nu^*(2H) - E$, therefore it is the blowing up of the conic in the intersection of Q_4 with another quadric which contains the conic.
FlF is the n. 16 in the Mori-Mukai list of Fano 3-folds with $Pic = 2$.

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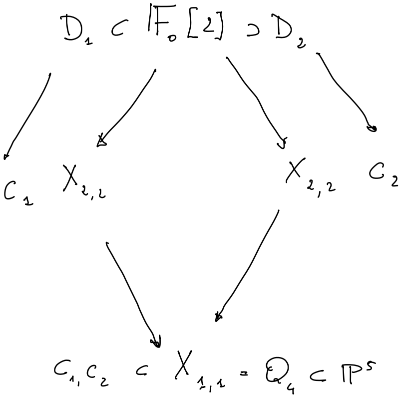
FlF is the n. 16 in the Mori-Mukai list of Fano 3-folds with $Pic = 2$.

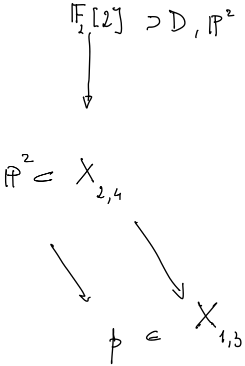
One can show that any FlF is also an hyperplane section of a $X_{1,3}$: take two general quadrics Q_1, Q_2 in $\mathbb{P}^5$, let C a smooth conic in the complete intersection $Q_1 \cap Q_2$ and Π a plane containing C.
In the pencil generated by the two quadrics there is a unique quadric Q_0 containing Π.

Birational geometry between $X_{2,2}$



"Sarkisov link"

Birational geometry between $X_{a,r-a}$



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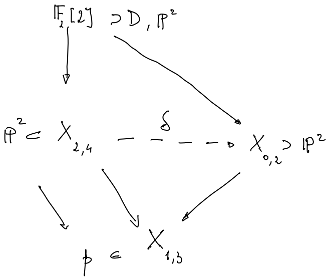

δ is a Atiyah flop

$X_{0,2}$ is obtained as the secant variety of S_0 (quadric cone)

$\cup$

$\mathbb{P}^2$ the secants through the singular point

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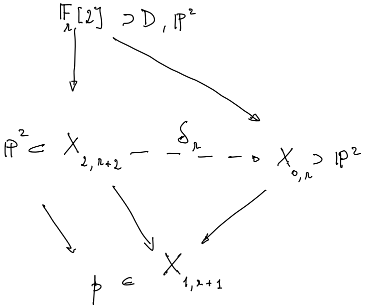

$X_{0,r}$ is the secant variety of S_r
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$\cup$
$\mathbb{P}^2$ secants through the singular point

$r = 1$ Kawamata small contraction and flip

Stability

We have some results on the G-stability of hyperplane section of $X_{1,3}$ (FlF) where G is the group of automorphisms coming from the one of $S_{1,3}$.

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We ask for the K-stability of FlF.