Lifting from an ample section
the case of weighted blow-ups

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Fano's 3-folds

Let $X \subset \mathbb{P}^N$ be a projective 3-fold such that for general hyperplanes H_1, H_2 the curve $\Gamma := X \cap H_1 \cap H_2$ is canonically embedded (i.e. K_Γ embeds Γ).

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Example: the quartic 3-fold in $\mathbb{P}^4$, $X_4 \subset \mathbb{P}^4$.

Fano Varieties and Fano-Mori Contractions

Definition

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If $Pic(X/Y) =< L >$ and $-K_X = rL$, r is called the *index of* X.

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Definition

Let $f : X \to Y$ be a contraction, X smooth or with mild singularities;
f is called a *Fano-Mori contraction* (F-M for short) if $-K_X$ is f-ample.
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A Fano manifold is a Fano-Mori contraction with $dimY = 0$.
A general fiber of a Fano-Mori contraction is a Fano variety.

Extremal rays

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f can be:
of fiber type ($dimX > dimY$), a Mori fiber space,
or birational, more precisely either divisorial or small.

Weighted Projective Space

Let $\sigma = (a_1, \dots, a_n) \in \mathbb{N}^n$ such that $a_i > 0$ and $\gcd(a_1, \dots, a_n) = 1$; let $M = \operatorname{lcm}(a_1, \dots, a_n)$.

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The weighted projective space with weight $(a_1, \ldots, a_n)$, denoted by $\mathbb{P}(a_1, \ldots, a_n)$, can be defined either as:

$$\mathbb{P}(a_1, \ldots, a_n) := (\mathbb{C}^n - \{0\})/\mathbb{C}^*,$$

where $\xi \in \mathbb{C}^*$ acts by $\xi(x_1, ..., x_n) = (\xi^{a_1} x_1, ..., \xi^{a_n} x_n)$.

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Or as:

$$\mathbb{P}(a_1, \dots, a_n) := Proj_{\mathbb{C}} \mathbb{C}[x_1, ..., x_n],$$

where $\mathbb{C}[x_1, ..., x_n]$ is the polynomial algebra over $\mathbb{C}$ graded by the condition $deg(x_i) = a_i$, for $i = 1, ..., n$.

Cyclic quotient singularities

A *cyclic quotient singularity*,

$$X := \mathbb{C}^n/\mathbb{Z}_m(a_1, \ldots, a_n),$$

is an affine variety definite as the quotient of $\mathbb{C}^n$ by the action $\epsilon : (x_1, \ldots, x_n) \to (\epsilon^{a_1} x_1, \ldots, \epsilon^{a_n} x_n)$, where ϵ is a primitive m-th root of unity.

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Let $Q \in Y : (g = 0) \subset \mathbb{C}^{n+1}$ be a hypersurface singularity with a $\mathbb{Z}_m$ action. The point $P \in Y/\mathbb{Z}_m := X$ is called a *hyperquotient singularity*.

Weighted Blow-up

Let $X = \mathbb{C}^n/\mathbb{Z}_m(a_1, ..., a_n)$ be a cyclic quotient singularity and consider the rational map

$$\varphi : X \to \mathbb{P}(a_1, \ldots, a_n)$$

given by $(x_1, \ldots, x_n) \mapsto (x_1 : \ldots : x_n)$.

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Definition

The *weighted blow-up* of $X = \mathbb{C}^n/\mathbb{Z}_m(a_1, ..., a_n)$ with weight $\sigma = (a_1, ..., a_n)$ (or simply the σ-blow-up), $\overline{X}$, is defined as the closure in $X \times \mathbb{P}(a_1, \ldots, a_k)$ of the graph of φ, together with the morphism $\pi_\sigma : \overline{X} \to X$ given by the projection on the first factor.

Weighted Blow-up

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Namely, let $e_i = (0, ..., 1, ..., 0)$ for $i = 1, ..., n$ and $e = 1/m(a_1, ..., a_n)$. X is the toric variety which corresponds to the lattice $\mathbb{Z}e_1 + ... + \mathbb{Z}e_n + \mathbb{Z}e$ and the cone $C(X) = \mathbb{Q}_+e_1 + ... + \mathbb{Q}_+e_n$ in $\mathbb{Q}^n$.

$\pi_\sigma : \overline{X} \to X$ is the proper birational morphism from the normal toric variety $\overline{X}$ corresponding to the cone decomposition of $C(X)$ consisting of $C_i = \Sigma_{j \neq i}\mathbb{Q}_+e_j + \mathbb{Q}_+e$, for $i = 1, ..., n$, and their intersections.

Weighted Blow-up

Consider now the function

$$\sigma\text{-wt} : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{Q}$$

defined on a monomial $M = x_1^{s_1} \dots x_n^{s_n}$ as $\sigma\text{-wt}(M) := \sum_{i=1}^{n} s_i a_i / m$.

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For any $d \in \mathbb{N}$ we define the σ-weighted ideal of degree d as

$$I_{\sigma,d} = \{g \in \mathbb{C}[x_1, \dots, x_n] : \sigma\text{-wt}(g) \geq d\} = (x_1^{s_1} \cdots x_n^{s_n} : \sum_{j=1}^{n} s_j a_j / m \geq d).$$

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Proposition

The weighted blow-up of of $X = \mathbb{C}^n/\mathbb{Z}_m(a_1, ..., a_n)$ with weight σ, $\pi : \overline{X} \to X$, is given by

$$\overline{X} = \mathrm{Proj}_X(\bigoplus_{d \geq 0} I_{\sigma,d}) \to X.$$

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Definition

Let $X : ((g = 0) \subset \mathbb{C}^{n+1})/\mathbb{Z}_m(a_0, ..., a_n)$ be a hyperquotient singularity and let $\pi : \overline{\mathbb{C}^{n+1}/\mathbb{Z}_m(a_0, ..., a_n)} \to \mathbb{C}^{n+1}/\mathbb{Z}_m(a_0, ..., a_n)$ be the $\sigma = (a_0, ..., a_n)$-blow-up.
Let $\overline{X}$ be the proper transform of X via π and call again, by abuse, π its restriction to $\overline{X}$.
Then $\pi : \overline{X} \to X$ is also called the *weighted blow-up of* X *with weight* $\sigma = (a_1, ..., a_n)$ (or simply the σ-blow-up).

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Proposition

Let $X : ((g=0) \subset \mathbb{C}^{n+1})/\mathbb{Z}_m(a_0, ..., a_n)$ be a hyperquotient singularity and let $i : X \to \mathbb{C}^{n+1}/\mathbb{Z}_m(a_0, ..., a_n)$ be the inclusion.
Then

$$\overline{X} = \mathbb{P}_X\Big(\mathcal{O}_X \oplus \bigoplus_{d \in \mathbb{N}, d>0} J^\sigma(db)\Big) \to X,$$

where $J^\sigma(db) := i^{-1}\big(I^\sigma(db)\big).\mathcal{O}_X$.

Lifting WBU

The technical result of our approach:

Theorem

Let $f : X \to Z$ be a local, projective, $\mathbb{Q}$-factorial contraction, which contracts an irreducible divisor E to an isolated $\mathbb{Q}$-factorial singularity $P \in Z$.
Assume that $dimX \geq 4$.

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Let $Y \subset X$ be a f-ample Cartier divisor such that $f' = f_{|Y} : Y \to f(Y) = W$ is a $\sigma' = (a_1, \ldots, a_{n-1})$-blow-up.

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The Theorem could be seen as a generalization of a paper of S. Mori: *On a generalization of complete intersections*, J. Math. Kyoto Univ., 1975.

Proof Part I: Lifting CQS

Proposition

Let Z be an affine variety of dimension $n \geq 4$ with an isolated $\mathbb{Q}$-factorial singularity at $P \in Z$.

Assume that $(W, P) \subset (Z, P)$ (germs of complex spaces around P) is a Weil divisor which is a cyclic quotient singularity, i.e. $W = \mathbb{C}^{n-1}/\mathbb{Z}_m(a_1, \dots, a_{n-1})$.

Then Z is a cyclic quotient singularity, i.e. $Z = \mathbb{C}^n/\mathbb{Z}_m(a_1, \dots, a_{n-1}, a_n)$.

Proof Part I

If W is a Cartier divisor, i.e. W is given as a zero locus of a regular function f, $W : (f = 0) \subset Z$, consider the map $f : Z \to \mathbb{C}$.
It is is flat, since $\dim_{\mathbb{C}} \mathbb{C} = 1$.
Quotient singularities of dimension bigger or equal then three are rigid, by a fundamental theorem of M. Schlessinger. Since Z has an isolated singularity and $dimW = n - 1 \geq 3$, it implies that W is smooth, i.e. $m = 1$. A variety containing a smooth Cartier divisor is smooth along it, therefore, eventually shrinking around P, Z is also smooth.

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In the general case we use Reid's trick: since Z is $\mathbb{Q}$-factorial, we can assume that there exists a minimal positive integer r such that rW is Cartier (r is the index of W). Take a Galois cover $\pi : Z' \to Z$, with group $\mathbb{Z}_r$, such that Z' is normal, π is etale over $Z \setminus P$, $\pi^{-1}(P) =: Q$ is a single point and the $\mathbb{Q}$-divisor $\pi^* W := W'$ is Cartier, $W' : (f' = 0) \subset Z'$.

Proof Part II, paraphrasing Mori

Let b a positive integer such that $-bE \sim Y$ (and therefore Cartier). By Grothendieck theory $X = \mathrm{Proj}_{\mathcal{O}_Z}(\oplus_{d\geq 0} f_*(\mathcal{O}_X(-dbE))$, therefore we want to prove that

$$f_*(\mathcal{O}_X(-dbE) = I_{\sigma,db} = (x_1^{s_1}\cdots x_n^{s_n} \mid \sum_{j=1}^{n} a_j s_j \geq db).$$

II part, paraphrasing Mori

Consider the exact sequence on X

$$0 \to \mathcal{O}_X(-Y - dbE) \to \mathcal{O}_X(-dbE) \to \mathcal{O}_Y(-dbE) \to 0$$

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Pushing it down via φ and using the relative Vanishing theorems we have

$$0 \to f_*\mathcal{O}_X(-(d-1)bE) \xrightarrow{\cdot x_n} f_*\mathcal{O}_X(-dbE) \to f_*\mathcal{O}_Y(-dbE) \to 0.$$

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The proposition follows by induction on n

$$(f_*(\mathcal{O}_Y(-dbE) = (x_1^{s_1} \cdots x_{(n-1)}^{s_{(n-1)}} \mid \sum_{j=1}^{n-1} a_j s_j \geq db)))$$

and on d

$$(f_*(\mathcal{O}_X(-(d-1)bE) = (x_1^{s_1} \cdots x_n^{s_n} \mid \sum_{j=1}^{n} a_j s_j \geq (d-1)b))$$

Counterexample in dimension 3, I part

Remark

If $n = 3$ we have the following example.

Example

Let $Z' = \mathbb{C}^4/\mathbb{Z}_r(a, -a, 1, 0)$; let (x, y, z, t) be coordinates in $\mathbb{C}^4$ and assume $(a, r) = 1$. Let $Z \subset Z'$ be the hypersurface given as the zero set of the function $f := xy + z^{rm} + t^n$, with $m \geq 1$ and $n \geq 2$.
This is a terminal singularity which is not a cyclic quotient (it is a terminal hyperquotient singularity).

However the surface $W := Z \cap (t = 0)$, which is the surface in $\mathbb{C}^3/\mathbb{Z}_r(a, -a, 1)$ given as the zero set of $(xy + z^{rm})$, is a cyclic quotient singularity of the type $\mathbb{C}^2/\mathbb{Z}_{r^2m}(a, rm - a)$.

Counterexample in dimension 3,

Example

Let $X = \mathbb{P}^2 \times \mathbb{P}^1$
Z be a $\mathbb{F}_1$ surface in the linear system $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^1}(1,1)$;

the contraction of the (-1) curve of Z lifts to the $\mathbb{P}^1$-bundle contraction onto $\mathbb{P}^2$.

The proof consists in lifting up the Cox Ring of the ample section to the variety.
The Cox Rings of weighted blow-ups determines completely the blow-up; this is true for any Toric variety.

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Question

Does the Cox Ring of an ample section determine the one of the variety?

Minimal Model Program- BCHM

Consider a log pair (X, Δ), i.e a normal variety X and an effective $\mathbb{R}$ divisor Δ, which is Kawamata log terminal (klt)
(that is $K_X + \Delta$ is $\mathbb{R}$-Cartier and for a (any) log resolution $g : Y \to X$ we have $g^*(K_X + \Delta) = K_Y + \Sigma b_i \Gamma_i$ with $b_i < 1$, for all i).

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If Δ is big by BCHM on a klt log pair (X, Δ) we can run a

$K_X + \Delta$- Minimal Model Program with scaling:

$$(X_0, \Delta_0) = (X, \Delta) \to (X_1, \Delta_1) \to - - -- \to (X_s, \Delta_s)$$

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1) (X_i, Δ_i) is a klt log pair, for $i = 0, ..., s$;

2) $\varphi_i : X_i \to X_{i+1}$ is a birational map which is either a divisorial contraction or a flip associated with an extremal ray $R_i = \mathbb{R}^+[C_i]$ such that $(K_{X_i} + \Delta_i) \cdot C_i < 0$
(notation: $R_i \in \overline{NE(X_i)}_{(K_{X_i}+\Delta_i)<0} \subset \overline{NE(X_i)}_{K_{X_i}<0}$)

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3) either $K_{X_s} + \Delta_s$ is nef (i.e. (X_s, Δ_s) is a log Minimal Model),
or $X_s \to Z$ is a Mori fiber space relatively to $K_{X_s} + \Delta_s$
(depending on the pseudeffectivity of $K_X + \Delta$).

Fano-Mori contractions on 3-folds

Terminal singularities in dimension 3 were classified by S. Mori (Nagoya 1985), they consist of hyperquotient singularities whose associated hypersurfaces in $\mathbb{C}^4$ could be finetely listed.
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Fano-Mori contractions on 3-folds

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They are all weighted blow-ups of (particular) cyclic quotient or hyperquotient singularities and the following should be true:

Conjecture

The divisorial contractions to a point for a MMP in dimension 3 *are weighted blow-up of a specific list of hyperquotient singularities.*

MMP for a q.p. pair- Adjunction Theory

Let (X, L) be a polarized (or quasi-polarized) variety, i.e. X has terminal singularities and L is a Cartier ample (or nef and bib) divisor; let $r \in \mathbb{Q}^+$.

Lemma (zip L into a boundary). Since L is nef and big there exists an effective $\mathbb{Q}$-divisor Δ^r on X such that

$$rL \sim_{\mathbb{Q}} \Delta^r \quad \text{and} \quad (X, \Delta^r) \text{ is Kawamata log terminal.}$$

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Run a $K_X + \Delta^r$-MMP and get a birational klt pair (X_s, Δ^r_s) which is
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Remarks/Problems

- Beyond the existence of the MMP, it would be nice to have a "description" of each steps and eventually of the Mori fiber spaces.
- (X_s, Δ^r_s) is not necessarily an (r) q.p. pair, i.e. we do not have a priori a nef and big Cartier divisor L_s such that $rL_s \sim_{\mathbb{Q}} \Delta^r_s$.

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If $\dim F \leq r+2$ then there exists $X' \in |L|$ with "good" singularities (i.e. as in X), except for two cases in which $n = 3$, $\dim F = r+2$ and φ is of fiber type.

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- Any section of L on X' lifts to a section of L on X...

Castelnuovo-Kawakita

Theorem

Let $\varphi : X \to Y$ be a birational contraction in a $K_X + \Delta^{n-2}$-MMP (i.e. it is associated with an extremal ray on a q.p. pair such that $R = \mathbb{R}^+[C] \in \overline{NE(X)}_{(K_X+(n-2)L)<0} \subset \overline{NE(X)}_{K_X<0}$ and $L \cdot C > 0$).

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Definition

We call such φ a Castelnuovo-Kawakita contraction.

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Apply the "lifting of weighted blow-up" to conclude.

Divisorial contractions in the (n-3)-MMP

Theorem

Let X be a variety with $\mathbb{Q}$-factorial terminal singularities of dimension $n \geq 3$ and let $f : X \to Z$ be a local, projective, divisorial contraction which contracts a prime divisor E to an isolated $\mathbb{Q}$-factorial singularity $P \in Z$ such that $-(K_X + (n-3)L)$ is f-ample, for a f-ample Cartier divisor L on X.
Then $P \in Z$ is a hyperquotient singularity and f is a weighted blow-up.