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Editors

When Form Becomes Substance

Power of Gestures, Diagrammatical Intuition and Phenomenology of Space

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Which Came First, the Circle or the Wheel? From Idea ($\iota\delta\varepsilon\alpha$) to Concrete Construction

Marco Andreatta

Abstract Idea is a Greek world ($\iota\delta\varepsilon\alpha$) which means mental representation, rational scheme or mathematical figure; Plato and Galileo said that nature uses the language of mathematics, whose geometrical patterns come from experiences. But, as F. Enriques noted, in the formalization of these ideas "we should take into account not really the experience, but rather the demands of simplification of our mind, in which they are reflected".

Most of our activities, especially in modern time, consists in realizing concretely abstract mathematical concepts and theories, as in the title of this conference: when form becomes substance. Fab Labs (Fabrication Laboratories) are modern workshops in which everybody can use digital fabrication to create real objects from mathematical ideas, especially from those which came from the phenomenology of space.

In the last part I will briefly consider some recent results in higher dimensional algebraic geometric, which can be summarized in few diagrams. I will point out that some of these ideas and diagrams could be directly connected to biology and life sciences.

Keywords Mechanical linkage · Fab Lab · Minimal model program

Introduction

The Greek philosopher Plato in the "Allegory of the Cave" tells that we live in a cave and that we perceive by our senses only shadows coming from another realm, the realm of pure forms, of ideas. This allegory is at the base of our way of doing philosophy and science, in particular mathematics.

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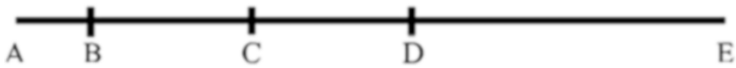
It comes immediately after the "Analogy of the Divided Line" (both contained in the Sixth book of the Republic): this is a nice diagrammatically explanation of how our brain interacts with the external world.

Shortly the analogy proposes to consider the visible things and the intelligible ones disposed on a line divided in two unequal parts, AC and CE. Then divide again each part in the same proportion, as in the figure: thus AB/AC = CD/CE = AC/AE.



The segments represent the following: AB the shadows and the reflections of physical things. BC the physical things themselves. CD mathematical reasoning, where abstract mathematical objects, such as geometric lines, are discussed. DE represents the subjects of philosophy, where we look at Ideas, which are given existence and truth by the Good itself. From the above proportion it follows easily that BC is equal to CD; it is not an accident that physical things corresponds to mathematical reasoning in Plato philosophy.

Idea is a Greek world ($\iota\delta\varepsilon\alpha$) which means scheme, pattern or mathematical figure. Our mind develops a process of knowledge based on this concept: a basic step consists in finding and giving an appropriate description of these ideas. After that, they must be connected to the physical things we perceive from nature, in a coherent and possibly effective way. This is a heavy task for mathematicians or, if one prefers, for philosophers, which is developing since the very beginning of our history.

In this process the use of diagrams, for instance of lines, circles, is fundamental, as Galileo often claimed. Note that the creation of an effective connection with nature comes through what we nowadays call technology.

Concerning the formalization of the ideas I find the following remark of F. Enriques very interesting: "we should take into account not really the experience, but rather the demands of simplification of our mind, in which they are reflected[1] . "In my opinion with this observation he stresses the fact that mathematics is an experimental science: starting from the observation of real things it makes use of the capacity of the brain to construct simplified models of them.

The evolution of human activity in the knowledge process had a critical point during the passage from epos to logos, that is from the oral tradition to the written ones. This passage started with Homer (seventh century b.C.) and was mature with Plato's writings. A big contribution to this transition was given by the works of Euclid, Archimedes, two mathematicians which wrote respectively the Elements and the Methods of Mechanical Theorems. In these books the intuitive power of diagram and the phenomenology of space start to have a written description.

[1] Federico Enriques, entry "Curve" in Enciclopedia Treccani.

I think that nowadays we are through a similar transition, from a written and oral tradition toward a digital tradition, both epos and logos have the chance to become less important. New digital instruments, both conceptual and technological, can, on one hand, easily reflect the natural demands of simplification of our mind and, on the other, make easy connections between the realm of ideas and concrete reality. In a very direct and "user friendly" way.

They are based on the Information Technology and its products, like web, google, wikipedia, robotic and so, ultimately, on mathematics.

Geometric Ideas and their Diagrams

Euclide's Elements (330 b.C), one of the most read and influent book in our history, is a basic step towards a written formalization of the mathematical ideas. In this Summa of the work of many mathematicians, like Thales, Pythagoras, Eudoxus, arithmetics and geometry become logos.

The first lines of the Elements contain some definitions and postulates:

Definitions.

1. A point is that which has no part
2. A line is breadthless length
3. A straight line is a line which lies evenly with the points on itself

...

15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another

Postulates.

1. To draw a straight line from any point to any point
2. To produce a finite straight line continuously in a straight line
3. To describe a circle with any center and radius
4. That all right angles equal one another
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

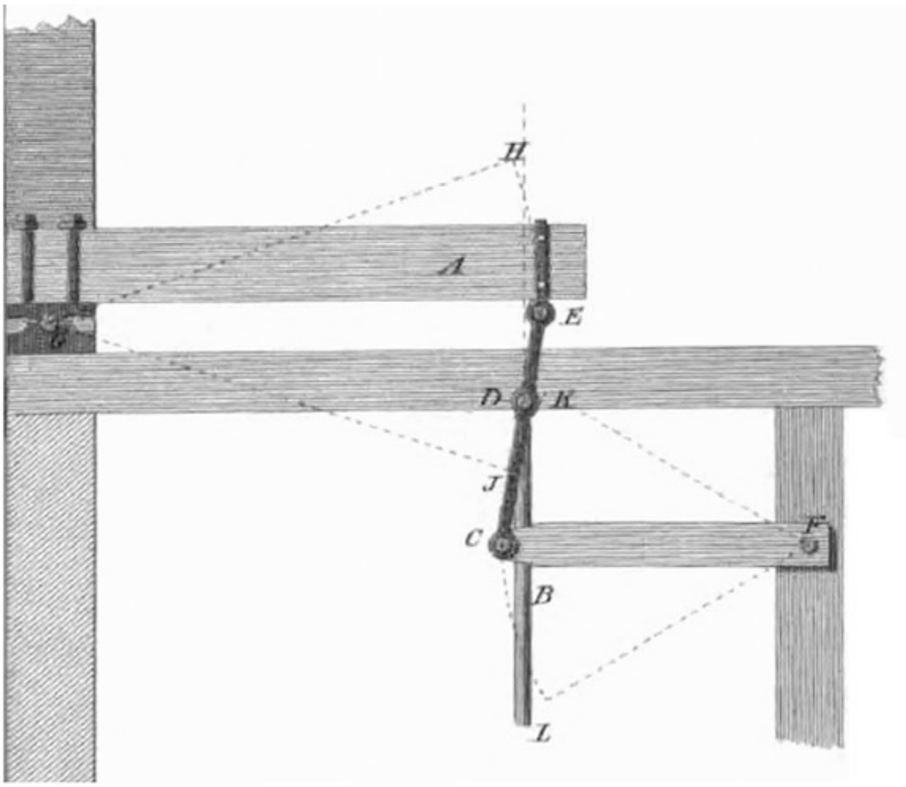
These few sentences give a coherent and ready to use definitions of two basic ideal diagrams, the line and the circle. They are a perfect translation in written words of two fundamental mathematical ideas, which can be find nowadays *verbatim* in any textbook all over the world. They are an "idealization" of real objects, like a wheel, the shape of the moon, the edge of the water at the horizon in a big see: they reflect the demands of simplification of our mind.

The fifth postulate is the famous one; it cannot be derived by the previous and it characterizes Euclidean geometry.

I’d like to concentrate on the postulates 1 and 3 which simply state that we can construct lines and circles. But, is it possible to construct effectively such “diagrams”?

Everybody can draw a circle: take a rope or a piece of an un-extendable material, fix an end of it and move around the other with a pencil attached. With the use of a proper *compass* we can vary the ray of the circle. According to the famous painting “Athen School” by Raffaello Sanzio (see picture),

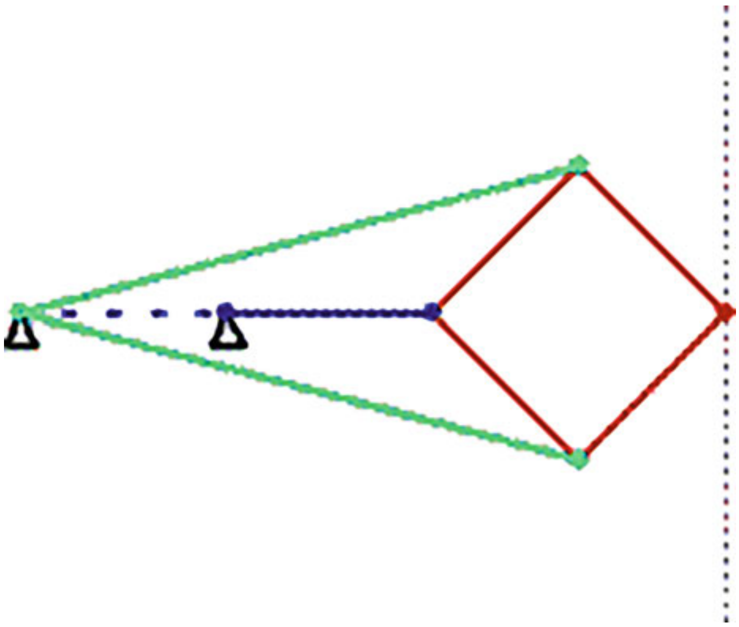
Euclid was using this instrument; there is actually a debate whether the use of compass occurs only later in the work of Arab mathematicians The question whether it is possible to draw a line with a compass, i.e. with a mechanical instrument, is much more difficult. In 1784 James Watt registered its patent specification for the now called Watt steam engine; in it he inserted a mechanism, called Watt’s linkage, with three bars, in which the central moving point of the linkage is constrained to travel on an approximation of a straight line. The following picture is the original design of Watt.



J. Watts wrote: “I have got a glimpse of a method of causing a piston rod to move up and down perpendicularly by only fixing it to a piece of iron upon the beam, without chains or perpendicular guides ... one of the most ingenious simple pieces of mechanics I have invented.” (letter to M. Boulton 1784).

This linkage does not generate a true line, and indeed Watt did not claim it. Rather, it traces out a Watt’s curve, an eight shaped figure called lemniscate. Nowadays this mechanism is also used in automobile suspensions, allowing the axle of a vehicle to travel vertically while preventing sideways motion.

Charles-Nicolas Peaucellier, a French army officer, constructed in 1864 the first mechanism capable of transforming rotary motion into perfect straight-line motion, called Peaucellier linkage. It consists of seven bars, four and two of the same size, connected as in the figure with two fixed points (marked with a triangle in the figure).



In the history of Mathematics we can find many mechanisms for the construction of curves, from Menaechmus, to Hippia, from Archimedes, to Descartes, Bernoulli, Huyghens and others.

Descartes dedicated much time to the effective construction of new compasses, among them the trisector, which draws a curve trisecting an angle, and the mesolabio, a curve giving the mean proportional of a segment. During these studies he developed a revolutionary theory to deal with curves, based on a new way of postulate the idea of curve which comes from Algebra.

In a letter to the mathematician Beckmann, dated 26.3.1619, he wrote: “So I hope I shall be able to demonstrate that certain problems involving continuous quantities can be solved only by means of straight lines or circles, while others can be solved only by means of curves produced by a single motion, such as the curves that can be drawn with the new compasses (which I think are just as exact and geometrical as those drawn with ordinary compasses), and others still can be solved only by means of curves generated by distinct independent motions which are surely only imaginary, such as the notorious quadratic curve [a curved line discovered by Hippias in the first century BCE; called ‘quadratix’ because it was used in attempts to square the circle.] With lines such as these available, I think, every imaginable problem can be solved. I’m hoping to demonstrate what sorts of problems can be solved exclusively in this or that way, so that almost nothing in geometry will remain to be discovered. This vast task is hardly suitable for one person; indeed, it’s an incredibly ambitious project. But I have glimpsed a ray of light through the confusing darkness of this science, and I think I’ll be able with its help to dispel even the thickest obscurities”.

The ray of light glimpsed by Descartes can be roughly summarized as follows. Starts with a diagram in a plane consisting of two perpendicular lines meeting in a point, called O; associate to each point in the plane a pair of numbers called coordinates, (x, y). Nowadays we call such construction a Cartesian coordinate system.

A plane curve is given by an equation in two variables, $f(x, y)$; its support is the set of points whose coordinates (x, y) satisfy the equation $f(x, y) = 0$. For instance, a circle centered in O with ray of length $r > 0$ is the set of points whose coordinates satisfies the equation $x^2 + y^2 = r^2$.

Let us read Descartes own words in the book *La Géométrie* (1637): "I could give here several other ways of tracing and conceiving a series of curved lines, each curve more complex than any preceding one, but I think the best way to group together all such curves and then classify them in order, is by recognizing the fact that all the points of those curves which we may call "geometric," that is, those which admit of precise and exact measurement, must bear a definite relation to all points of a straight line, and that this relation must be expressed means of a single equation.*"*[2]

This great idea of Descartes connects geometry to algebra but at that time created some troubles to philosophers: in fact it opposes some theories of Aristotle, who wrote in the book Posterior Analytics: "a proof in one science cannot be simply transferred to another, e.g. geometric truth cannot be proven *arithmetically"*.

This astonishing change of perspective was made possible by the many results in algebra obtained by Italian mathematicians in the previous century. It was in the air and a similar theory was developed independently by the great competitor of Descartes, Pierre Fermat. Nowadays we call this theory Analytic or Algebraic Geometry and it is one of the central area of research in modern Geometry.

Descartes then showed that all classical curves could be described by equations in his new method; he proved for instance that conics are given by polynomials of degree two and that these polynomials give no other curves (a part pairs of straight lines).

Looking at polynomials of higher order he founded new curves, among them for instance the folium, with equation $x^3 + y^3 = 3axy$ (or $x(t) = \frac{3at}{1+t^3}$, $y(t) = \frac{3at^2}{1+t^3}$ where t is a parameter).

This curve has a singular point (a node) at the origin and Descartes was not able to draw it, thinking that it repeats equally in the four quadrants; Christiaan Huygens gave a first exact drawn of it. The problem of finding the tangent to this curve started a famous quarrel between Descartes and Fermat.

In this set up a big difference is to consider curves whose equation is given by a polynomial or those defined by a transcendental or analytic equation. Descartes himself made a distinction between two classes of curves, namely the admissible or geometric ones and the mechanicals or imaginaries; the quadratrix of Hippias mentioned above is among the imaginaries, for example.

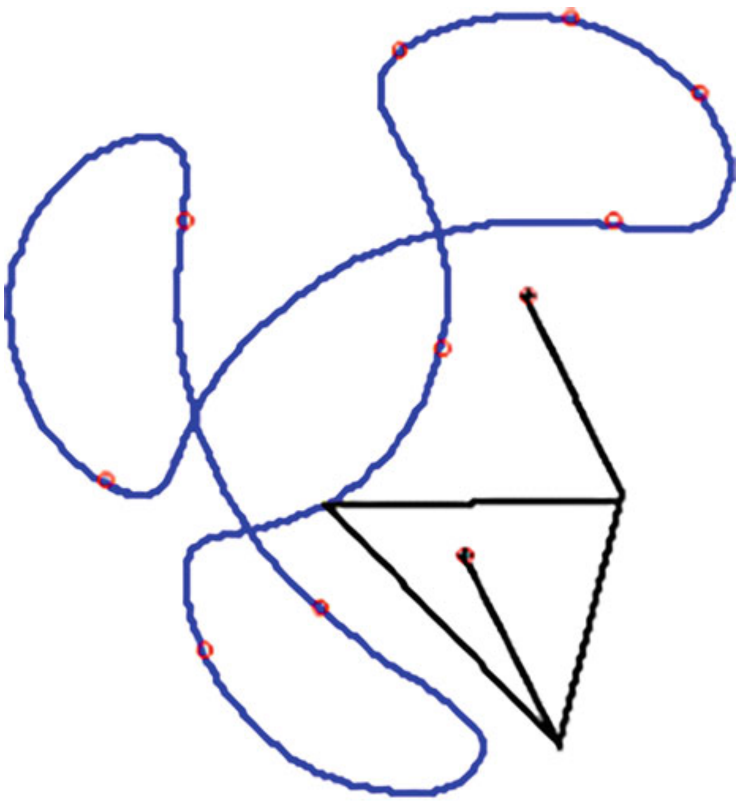
Much later, in 1876, A.B. Kempe[3] proved a fundamental Theorem, following which a plane curve given as zero of a Polynomial of two variables can be drawn by a mechanical linkage.

[2] Descartes *La Géométrie*, (1637).

[3] Kempe (1875), p. 213–216.

This clarifies definitely the problem of when a curve in the plane can be drawn with a “new compass”.

On the other hand, it opens the problem to construct effectively a compass to draw a given curve or a diagram. These problems have been extensively studied by engineers, let me mention for instance the following one: given 9 points in the plane find a Four-bar linkage which draws an algebraic curve passing through the nine points. The existence of the algebraic curve (and of the linkage) is mathematically clear (for more points there could be no solutions). The problem is to find it in a reasonable time; General Motor some years ago asked for a solution in order to create an optimal windscreen wiper. In 1992 the mathematician A. Sommese and others proved that given the nine points one can reduce to 1442 possible linkage; a check in this finite a priori list easily gives the correct ones. More recently other mathematicians reduced the a-priori possibilities to the number 64. The next figure represents a four bars linkage which draws a curve passing from nine given points.

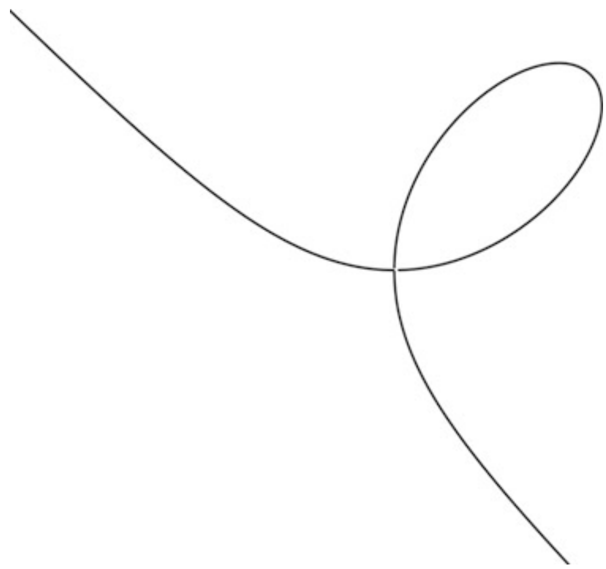


Digital Fabrication

The use of curves or diagrams to describe phenomena in real life and to solve various problems, as indicated by Greek mathematicians and re-proposed by Descartes, is nowadays a general method in all sciences.

Curves are defined *a la Descartes* with the use of equations and they can be drawn with the help of computers. In my opinion this fact is really a big step in the contemporary culture: anybody can easily draw a curve via its equation using a free and user-friendly software, for instance the one denominated GeoGebra.[4]

I still remember the difficulties I found in the high school when the teacher asked to draw a parabol or an ellipse of a given equation. Even Descartes was fouled up by drawing the folium; the following drawn was done in few second simply writing the equation in the low line of a GeoGebra Sheet.

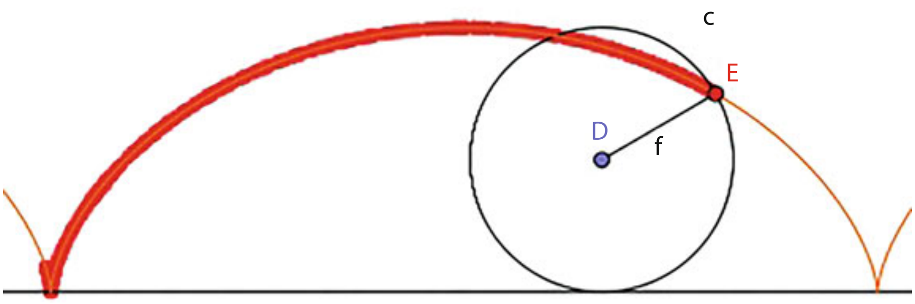


An amusing experience could be to go through the book *La Géométrie*, performing the constructions of Descartes with GeoGebra[3]. This has been actually done recently in a master thesis by Sara Gobbi,[5] a student of the University of Trento. It is surprising to notice, on one hand, how effective is the Descartes language to talk virtually with digital reality; and, on the other hand, how much the digital technology can help in understanding deep ideas of Analytic Geometry. I do not enter here in the question of the approximation, needed by computers to reproduce an abstract curve via numerical analysis.

With this software it is also very easy to construct digitally mechanisms and new compasses which draw curves; the following figure is the drawing of a cycloid made with GeoGebra with a circle rotating on a line.

[4] GeoGebra: https://www.geogebra.org/

[5] S. Gobbi Master Thesis, University of Trento: https://www.geogebra.org/m/azTYkAfn



The cycloid is a famous curve which is either a Brachistochrone and a Tautochrone; curves with such properties were searched by Galileo and found by Bernoulli, Huyghens, Leibniz, Newton and others.

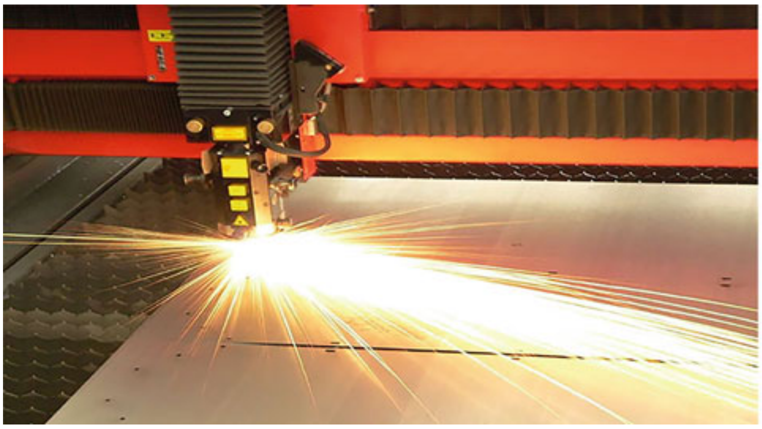
Digital visualization of curves may not be such a great novelty for many of us. What is newer is that these digital objects can now “come out of” the computers and become true border shapes. For that one should simply visit a Fab Lab, i.e. a Fabrication Laboratory, a small-scale workshop offering (personal) digital fabrication. These laboratories have many tools which are directly controlled by computers, among others laser cutters (for glass, plastic, metal, wood, ...) and 3D-printers (printing in plastic, gold, chocolate, ...).

The first Fab Lab seems to be the one created in 2001 at MIT in US. Now, any town of the industrialized world as a Fab Lab and many cultural or scientific institutions host them. On December 2017 about 1200 Fab Labs were officially listed on a web page of the Fab Lab community; on 2017 a Fab Lab based in Grenoble was vandalized and burned by anarchists.

The following are two pictures of the Fab Lab of MUSE, the Science Museum of Trento-Italy.[6]

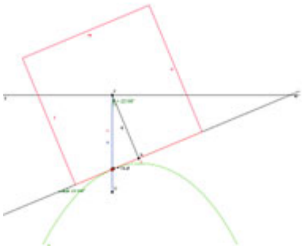
The files constructed with GeoGebra (or other software) can be easily “read” and interpreted by the laser cutters.

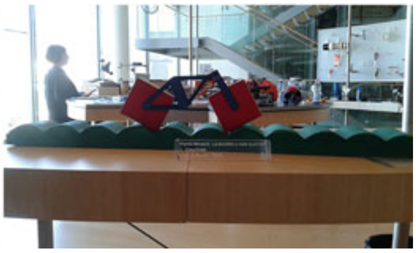
[6] Fabrication Laboratory at MUSE-Trento, Italy: http://fablab.muse.it/



For instance, one can cut a cycloid in the wood (adding a line one produces an exhibit which gives a nice tool to show the brachistochrone property of the curve):

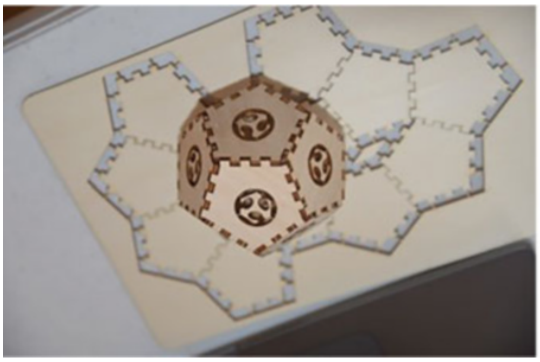
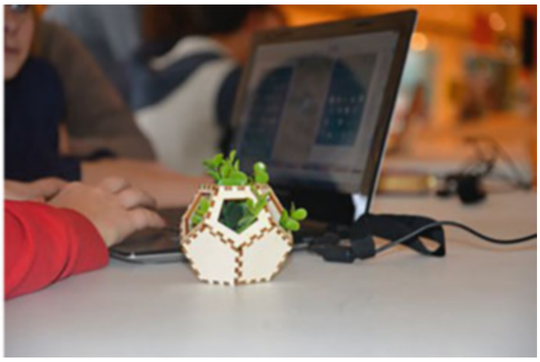
The following pictures represent a project to construct a bicycle with square wheels. In the first one, with GeoGebra, the profile of a suitable ground has been constructed, a catenary. A proof has been made in wood and finally a bike, with the appropriate ground, has been constructed by a school: this project was the winner of a national school competition called Premio Bonaccina in 2017.





These are simple examples related of the use of a laser cutter to produce objects interesting from the point of view of pure mathematics and its popularization. The principal goal of a Fab Lab is to create economic development for the society in which it acts; the Fab Lab of MUSE hosts everyday public and private enterprises which construct many prototypes of objects they would like to produce.

I like to provide two other examples related to arts: the first is the realization of a flowerpot shaped as a dodecahedron.

The second are two example of “String Art”, modeled on the famous Naum Gabo’s operas, realized by the artist David Press for the bookshop of the Momath at New York; the plastic support and the holes are cut out with a laser cutter.

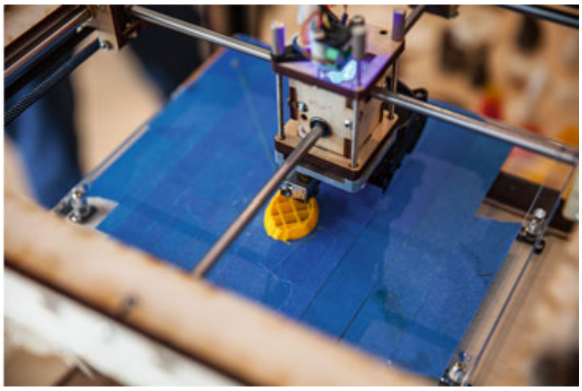
The definition of a curve as the locus of zero of a function in two variables was extended to the case of surfaces in 1700 by A. Parent: for him a surface is the set of points in the space whose coordinates $(x,\ y,\ z)$ have a realization given by an equation $f(x,\ y,\ z) = 0$ where f is a function, for instance a polynomial, in three variables.

Given such a function it is usually rather difficult to figure out what is the shape of the associated surface. Digital visualization on computer has been largely developed in these years, mainly to produce smart video games or animation movies. A very simple and free software which I suggest in order to visualize surfaces associated to algebraic functions is called Surfer, it can be downloaded from the web site imaginary.org.[7]

Here are some examples, the first two represent the hyperbolic hyperboloid $(ax^2 + b\ y^2 - c\ z^2 - 1)$ and the hyperbolic paraboloid $(ax^2 - b\ y^2 - c\ z)$, the two ruled quadrics.

[7] Imaginary Open Mathematics: https://imaginary.org/

The files produced with Surfer (or any other software include the CAD and CAM ones developed by engineers and architects) can be interpreted by a 3D-printers in a FabLab.



In principle one can create any ideal surfaces with many type of material: the possibilities actually depend very much on the quality (and thus on the prize) of the 3D-printer and on the skill of the Fab Lab technician responsible of the printer. Those people face the same problems that Benvenuto Cellini had in producing his famous Saliere and Statues, like for instance the Perseus in Florence. The surface could not usually be printed in one piece but in several ones, which then should be glued together; this job requires good skills and experience.

The following pictures give some examples: two famous surfaces, namely the surface of Dini, a surface with negative curvature in any point (a model of non-Euclidean plane geometry), and the surface of Barth, a surface of degree six with the maximum number of singularities. The ready to print files were realized by Oliver Lab and can be downloaded at the above quoted web site: imaginary.org.

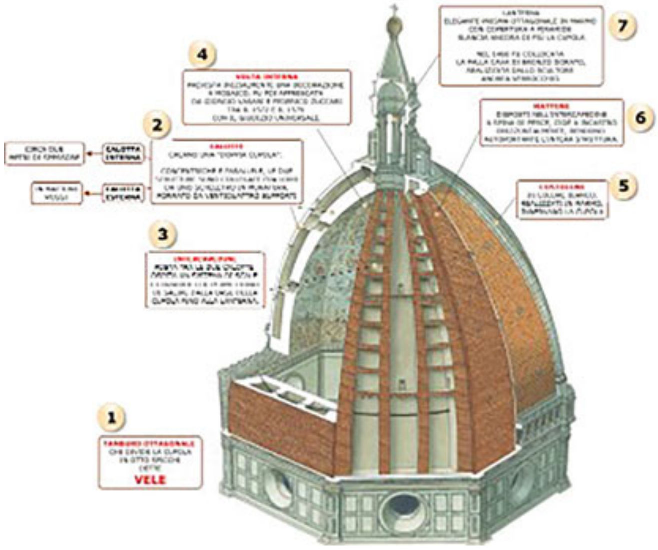
The amazing work of Oliver Lab, a "mathematicians with interests in design, programming and several aspects of mathematics", can be explored on his web pages.[8] There one can also buy many surfaces realized with the 3D-printers, artistic objects to display in a room or as jewellery to wear.

In the next three pictures one can see the same cubic surface: the first was constructed in wood by Campedelli in the 1951 and can be found in the collection of Museo della Scienza in Milano. The other two are realized with a 3D-printer by Oliver Lab, the last is a mathematical jewellery.



The famous Brunelleschi Dome of Florence was the biggest dome of the world, still the biggest constructed of masonry. Completed in 1436 with the technique of the "sixth of fifth or fourth of angle", has a catenary as curve section, as discovered by Bernoulli and Huygens in 1690. Together with some of my students of the University of Trento we figure out a digital representation of the Dome which we then produced with the 3D-printer.

[8] Oliver Lab, Mathematical Objects: https://oliverlabs.net/



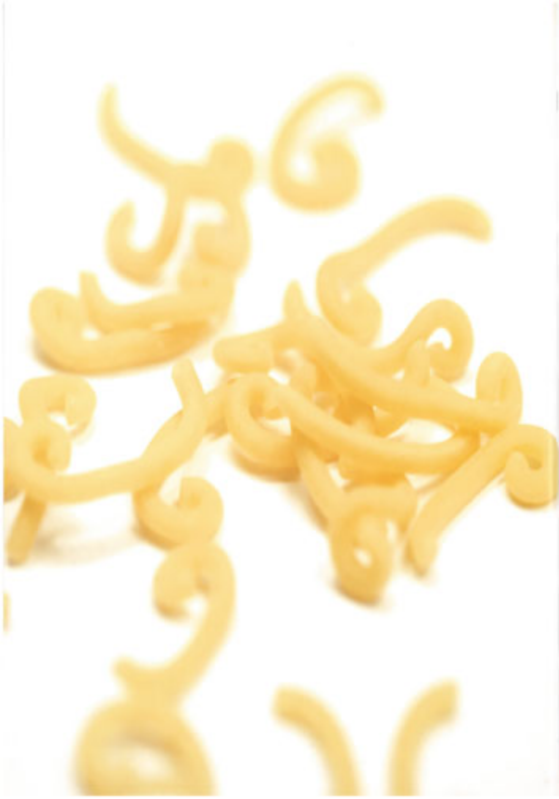
It is possible to construct full houses with appropriate 3D-printers, they can be ordered on the web.

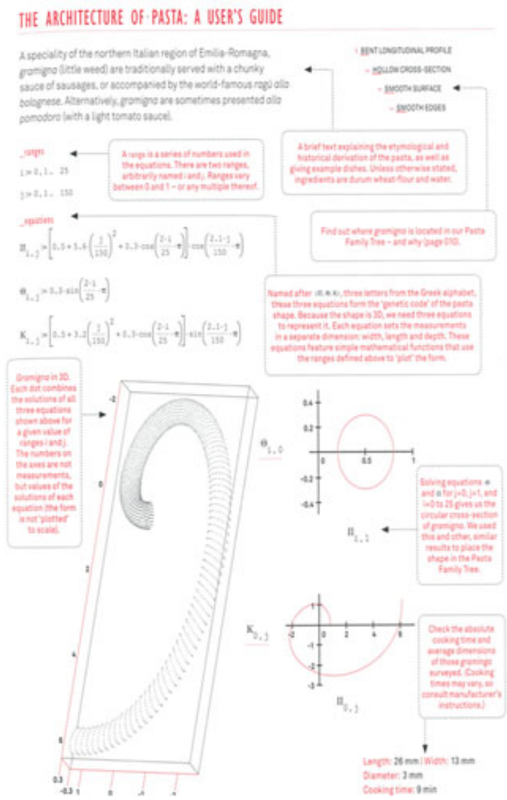
With a 3D-printer one can even try to create surfaces living in higher dimension, like the following Klein's bottle.

The shape of the food is an important feature of its enjoyment; the success of the Pringles chips depends also on their shape, a hyperbolic paraboloid.

Last year the Barilla Company, an Italian food enterprise, open a food contest for the creation of new pasta shapes to be realized through innovative pasta 3D-printer. Interested competitors should, in my opinion, consult the beautiful book Pasta by Design,[9] by architect George Legendre in collaboration with Paola Antonelli, curator of MoMA's Department of Architecture and Design in New York. The book offers a classification of pasta shapes through 92 "canonical models", morphologically different and connected within a phylogenetic tree, described through an equation in three-dimensional space and, finally, associated with a specific sauce; the next figure refers to two pages of the book.

[9] George Legendre et oth. (2011).





Geometry in Higher Dimension

In the XX century Geometry started to explore more general spaces, which are not confined in the usual (standard) 3-dimensional ambient but described by more variables. The great breakthrough was obtained some years before through the work of Bernhard Riemann (1826–1866). In his "Habilitation" thesis",[10] he overcame the limitations of the classical view and founded modern geometry, introducing n-dimensional varieties, varieties with curvature, the celebrated non- Euclidean geometries and the so-called Riemann surfaces.

Geometry in arbitrary high dimension requires a good capability of abstraction, the objects here are very much ideal. In particular it is hard "to visualize" them and we need new concepts for constructability. The use of diagrams is turning out to be very essential for these purposes, as well as new digital techniques.

In the fall of 2016, I was a curator for an exhibition, at MUSE-Trento-IT, dedicated to mathematics. In one section we considered the question of visualizing higher dimensional objects: for example, to visualize regular 4-dimensional polytopes we described their projections via 3-dimensional diagrams and digital visual construction. The next pictures refer to some of those exhibits dedicated to the polytope denominated "120 cells".

[10] Bernhard Riemann (1867)

The digital constructions were realized by Gian Marco Todesco and his Digital Video s.p.a.: a sample can be find on YouTube.[11]

Higher dimensional varieties are deeply studied in Algebraic Geometry; the theory was developed in the XXth century by the work of many mathematicians, including the Italians, like Enriques, Castelnuovo, Severi, Segre, Fano. and French ones, like Serre, Weil, Grothendieck.

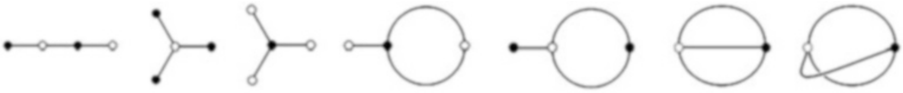
Alexander Grothendieck in particular introduced the language of Schemes, which clarifies in term of algebra many subtle facts about the reciprocal relations between geometric objects: intersection, multiple or non-reduced structure, ... The use of diagrams to describe algebro-geometric structure reached within this language an acme: examples are the Dynkin diagrams for algebras and singularities, polytopes for Toric Varieties, ...

Grothendieck, in his famous proposal for a long-term mathematical research "Equisse d'un Program" (Sketch of a Programme) (1984), introduced also some special diagrams which he called Dessins d'enfants. They are embedded graphs used to study Riemann surfaces and to provide invariants for the action of the absolute Galois group. A dessin d'enfant is the French term for a 'child's drawing' and A. Grothendieck commented his discovery with the following words: "this discovery, which is technically so simple, made a very strong impression on me, and it represents a decisive turning point in the course of my reflections, a shift in particular of my center of interest in mathematics, which suddenly found itself strongly focused. I do not believe that a mathematical fact has ever struck me quite so strongly as this one, nor had a comparable psychological impact. This is surely because of the very familiar, non-technical nature of the objects considered, of which any childs drawing scrawled on a bit of paper (at least if the drawing is made without lifting the pencil) gives a perfectly explicit example. To such a dessin we find associated subtle arithmetic invariants, which are completely turned topsy-turvy as soon as we add one more stroke."[12]

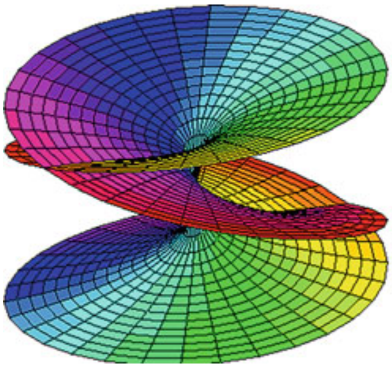
The next picture represents some dessins d'enfants associated to basic Riemann surface.

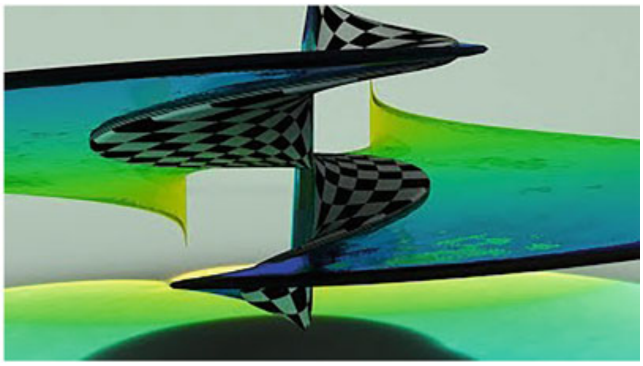
[11] Gian Marco Todesco video: https://www.youtube.com/watch?v=0uT6q_hrK50

[12] Grothendieck (1984).



A Riemann surface is actually a 2-dimensional object but it usually lives in a 4-dimensional space; in the next picture you can see a digital representation of a simple Riemann surface and a realization with a 3D-printer.

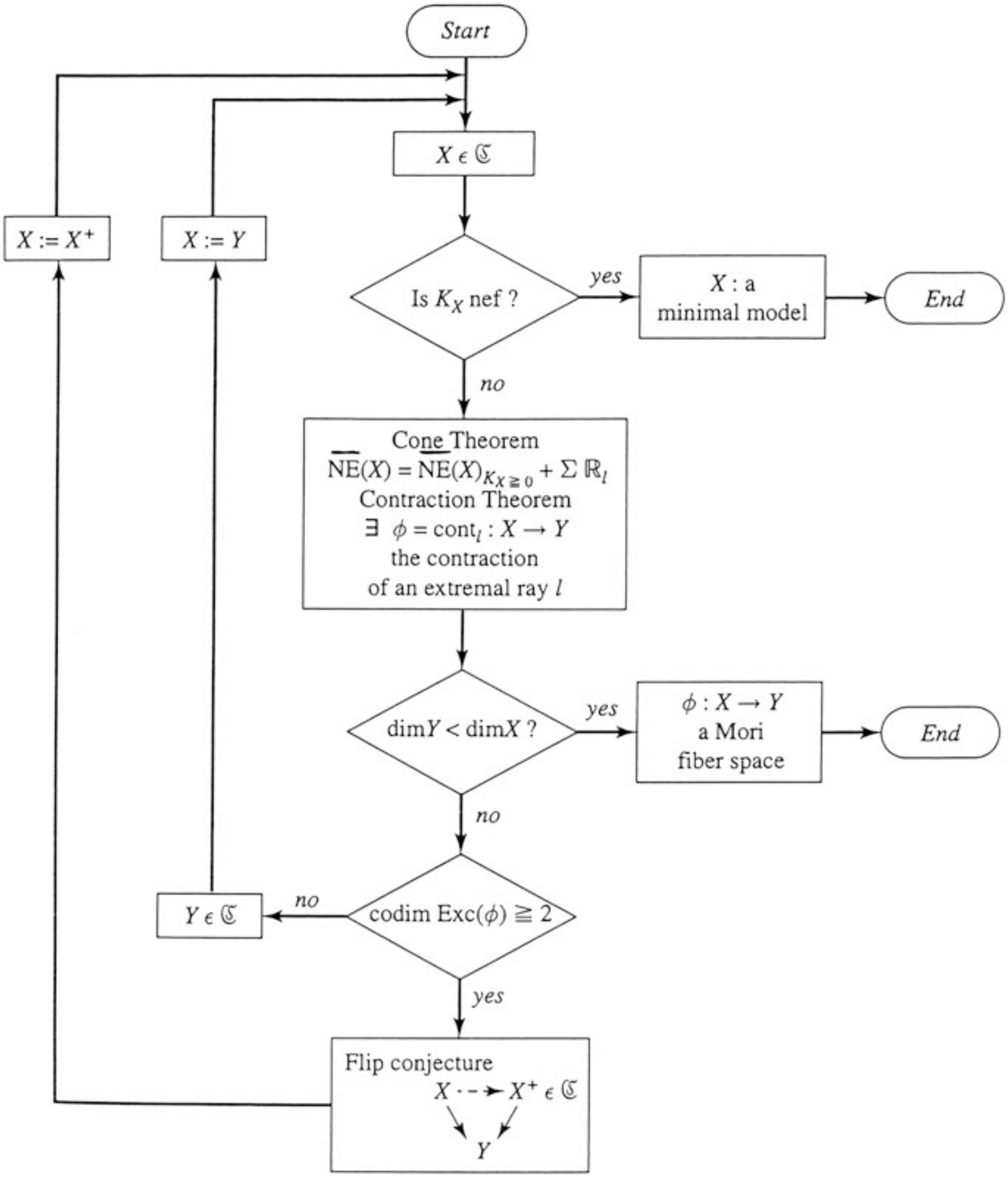




The classification of Algebraic Varieties embedded in a Projective space is a central topic in modern geometry. In 1980 the Japanese mathematician Shigefumi Mori suggested a Program to classify those varieties in every dimension. This program consists in two main steps: first one has to identify the so-called Special Minimal Models (roughly speaking, varieties that do not admit further projections). Secondly to find a general procedure that, starting from any variety, through projections and projective transformations, leads to a corresponding Minimal Model. The following diagram, taken from the book of K. Matsuki,[13] resumes the full Program and it is well known to specialist.

[13] Matsuki (2002).

Minimal Model Program in Dimension 3 or Higher (in Arbitrary Dimension)



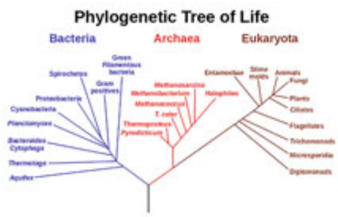
Mori himself has proved the feasibility of the Minimal Model Program in three dimensions; for this he was awarded the Fields Medal in 1990. In 2010, an international team of four mathematicians, Caucher Birkar, Paolo Cascini, Christopher D. Hacon and James McKernan, with a brilliant article in the Journal of the American Mathematical Society,[14] proved the feasibility of the program in any

[14] Birkar (2010), n. 2, p. 405–468.

dimension. Now the problem has shifted to find effective algorithms that determine a minimal model for a given variety.

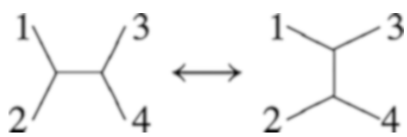
It turned out that the fundamental building blocks of the Program, like the atoms for the ordinary matter, aresomevarietiesintroduced acentury ago by the Italian mathematician Gino Fano.

A group of researchers, including Bernd Sturmfels, proposed an interesting connection between phylogenetic trees, which we find in the theory of biological evolution, and some algebraic projective varieties called Toric and Tropical varieties. In the following picture one can find the original draw of Darwin, a modern example of phylogenetic tree and an exhibit of MUSE on a "universal" phylogenetic tree.



My colleague J. Wisniewski, in this way, associated Fano varieties to trivalent trees. Subsequently, by applying the minimal model program to them, he obtained new examples of Fano varieties. A very nice discover in the field of geometry which starts from a biological diagram. On the other way around, at the moment there is not a good "biological" explanation of the result; I am sure that its importance in life science will be eventually discovered.

A fundamental step in the Minimal Model Program is a mathematical operation called a flop (invented by the Field medalist Sir M.F. Atyah long ago); in the Wisniewski construction on the phylogenetic tree it corresponds to the operation described by the following diagram.



I like to conclude with a nice record of the conference. In that period in Paris I had the opportunity to visit an exhibition on Edgard Degas at the Gare d'Orsay ("Degas Danse Dessin", January 2018). There I could read the following sentence of Paul Valery about Degas's work: "There is a huge difference between seeing something without a pencil in your hand and seeing it while drawing it. Or rather you are seeing two quite different things. Even the most familiar object becomes something else entirely, when you apply yourself to drawing it: you become aware that you did not know it-that you had never truly seen it . . . It dawns on me that I did not know what I knew: my best friend nose.[15]".

I find it very pertinent for the actual occasion: although we are not artists, when we draw a diagram, even digitally, very often we see a different thing, may be something we had never truly seen before.

Bibliography

F. Enriques, entry "Curve" in Enciclopedia Treccani
R. Desartes, "La Géométrie", 1637
A.B. Kempe: On a General Method of describing Plane Curves of the n-th degree by Linkwork, Proceedings of the London Mathematical Society, 1875
G. Legendre et oth., "Pasta by Design", 2011
A. Grothendieck, "Equisse d'un Program", 1984
K. Matsuki, Introduction to the Mori Program, Universitext Springer, 2002
C. Birkar, P. Cascini, C. D. Hacon, J. McKernan: Existence of minimal models for varieties of log general type. J. Amer. Math. Soc. 23, 2010
P. Valery, "Degas Danse Dessin", 1938

Sitography

App di Matematica GeoGebra: https://www.geogebra.org/
S. Gobbi, Master Thesis, University of Trento: https://www.geogebra.org/m/azTYkAfn
Fabrication Laboratory at MUSE-Trento, Italy: http://fablab.muse.it/
Imaginary Open Mathematics: https://imaginary.org/
Oliver Lab, Mathematical Objects: https://oliverlabs.net/
Bernhard Riemann, On the hypotheses which lie at the foundation of geometry, Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, vol. 13, 1867: https://www.maths.tcd.ie/pub/HistMath/People/Riemann/Geom/
Gian Marco Todesco, video: https://www.youtube.com/watch?v=0uT6q_hrK50

[15] Paul Valery (1938).