MORPHISMS OF PROJECTIVE VARIETIES FROM THE VIEWPOINT OF MINIMAL MODEL THEORY

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Introduction

One of the main result of last decades algebraic geometry was the foundation of Minimal Model Program and its proof in dimension three. Minimal Model Theory shed a new light on what is nowadays called higher dimensional geometry. In mathematics high numbers are really a matter of circumstances and here we mean greater or equal than 3. The impact of MMP has been felt in almost all areas of algebraic geometry. In particular the philosophy and some of the main new objects like Fano-Mori spaces and log varieties started to play around and give fruitful answer to different problems.

The aim of Minimal Model Program is to choose, inside of a birational class of varieties, "simple" objects. The first main breakthrough of the theory is the definition of these objects: Minimal Models and Mori Spaces. This is related to numerical properties of the intersection of the canonical class of a variety with effective cycles. After this, old objects, like the Kleiman cone of effective curves and (-1)-curves, acquire a new significance. New ones, like Fano-Mori contractions, start to play an important role. And the tools developed to tackle these problems allow the study of formerly untouchable varieties.

Riemann surfaces were classified, in the XIX^{th} century, according to the curvature of an holomorphic metric. Or in other words according to the Kodaira dimension. Surfaces needed a harder amount of work. For the first time birational modifications played an important role. The theory of (-1)-curves studied by the italian school of Castelnuovo, Enriques and Severi, at the beginning of XX^{th} century, allowed to define minimal surfaces. Then the first rough classification of the latter, again by Kodaira dimension was fulfilled. As we already said Minimal Model Program is now a tool to start investigate this question in dimension 3 or higher.

In writing these notes we want to give our point of view on this area of research. We are not trying to give a treatment of the whole subject. Very nice books appeared recently for this purpose, and we often refer to them in the paper. We would like to present, in a sufficiently self contained way, our contributions and interests in this field of mathematics. We will study Fano-Mori spaces both from the biregular and the birational point of view. For the former we will recall and develop Kawamata's Base Point Free technique and Mori's deformation arguments. For the latter we lean on Sarkisov and #-Minimal Model Programs.

The content of the parts is the following. In Part 1 we collect the main definitions and Theorems we are going to use afterwards.

Part 2 is studied to get the reader acquainted with the Base Point Free technique. For this purpose we give, or sketch, several proofs of Kawamata's Base Point Free Theorem, trying to hide the technicalities.

In Part 3 we introduce the main actor of the book, Fano-Mori contractions and more generally Fano-Mori spaces. Using BPF we describe then the main properties that will allow us to study them.

Part 4 is the applications of all the above to smooth varieties. Namely we give a biregular classification of Fano-Mori spaces of dimension less than or equal to four and Mukai manifolds.

In Part 5 we present the other side of the moon, the birational world. Here a beautiful old Theorem of Castelnuovo and Noether is proved in a modern language. Philosophy and applications of Minimal Model Program for threefolds are outlined.

These notes collect some topics we presented in three mini-courses which were held in Wykno (Pl) (1999), Recife (Br) (2000) and Ferrara (It) (2000), respectively. We discussed with many people about this subject and we are very thankful to them all. But we would like to distinguish Jaroslaw Wiśniewski and thank him deeply.

Part 1. Preliminaries

In this part we will collect all definitions which are more or less standard in the algebraic geometry realm in which we live.

1.1. The Kleiman–Mori cone of a projective variety.

First we fix a good category of objects (real differentiable varieties are not the good ones to extend the Riemann and Poincaré approach). Let X be a normal variety over an algebraically closed field k of dimension n, that is an integral separated scheme which is of finite type over k. We will actually assume also that $char(k) = 0$, nevertheless many results at the beginning of the theory hold also in the case of positive characteristic.

We have to introduce some basic objects on X.

Let $Div(X)$ the group of Cartier divisors on X and $Pic(X)$ be the group of line bundles on X. Let also $Z(X)$ be the group of Weil divisors and Z_1 be

the group of 1-cycles on X i.e. the free abelian group generated, respectively, by prime divisors, and reduced irreducible curves.

We will often use $\mathbb{Q}$-Cartier divisors, that is linear combinations with rational coefficients of Cartier divisors. For these objects it is useful to introduce the following notations. Let $D = \sum d_i D_i$ a $\mathbb{Q}$-Cartier divisor, then $\lfloor D \rfloor := \sum \lfloor d_i \rfloor D_i$, $\lceil D \rceil := -\lfloor -D \rfloor$ and $\langle D \rangle := D - \lfloor D \rfloor$, where $\lfloor d_i \rfloor$ is the integral part of d_i.

Then there is a pairing

$$Pic(X) \times Z_1(X) \to \mathbb{Z}$$

defined by $(L, C) \to LC := deg_C(L_{|C})$, and extended by linearity.

Two invertible sheaves $L_1, L_2 \in Pic(X)$ are *numerically equivalent*, denoted by $L_1 \equiv L_2$, if $L_1 C = L_2 C$ for every curve $C \subset X$. Similarly, two 1-cycles C_1, C_2 are *numerically equivalent*, $C_1 \equiv C_2$ if $LC_1 = LC_2$ for every $L \in Pic(X)$.

Define

$$N^1 X = (Pic(X)/\equiv) \otimes \mathbb{R} \text{ and } N_1 X = (Z_1(X)/\equiv) \otimes \mathbb{R};$$

obviously, by definition, $N^1(X)$ and $N_1(X)$ are dual $\mathbb{R}$-vector spaces and $\equiv$ is the smallest equivalence relation for which this holds.

In particular for any divisor $H \in Pic(X)$ we can view the class of H in $N^1(X)$ as a linear form on $N_1(X)$. We will use the following notation:

$$H_{\geq 0} := \{x \in N_1(X) : Hx \geq 0\} \text{ and similarly for } > 0, \leq 0, < 0$$

and

$$H^{\perp} := \{x \in N_1(X) : Hx = 0\}.$$

The fact that $\rho = dim_{\mathbb{R}} N^1(X)$ is finite is the Neron-Severi theorem; the natural number ρ is called the Picard number of the variety X. (Note that for a variety defined over $\mathbb{C}$ the finite dimensionality of $N_1(X)$ can be read from the fact that $N_1(X)$ is a subspace of $H_2(X, \mathbb{R})$).

More generally, given a projective morphism $f : X \to Y$ and $A, B \in Div(X) \otimes \mathbb{Q}$, then A is *f-numerically equivalent* to B ($A \equiv_f B$) if $A \cdot C = B \cdot C$ for any curve C contracted by f. A is *f-linearly* equivalent to B ($A \sim_f B$) if $A - B \sim f^* M$, for some line bundle $M \in Pic(Y)$, we will suppress the subscript when no confusion is likely to arise.

Note that if X is a surface then $N^1(X) = N_1(X)$; but, using M. Reid words (see [Re4]), "Although very simple, this is one of the key ideas of Mori theory, and came as a surprise to anyone who knew the theory of surfaces before 1980: the quadratic intersection form of the curves on a nonsingular surface can for most purpose be replaced by the bilinear pairing between N^1 and N_1, and in this form generalizes to singular varieties and to higher dimension."

We notice also that *algebraic equivalence* of 1-cycles implies numerical equivalence. Moreover, if X is a variety over $\mathbb{C}$ then, in terms of Hodge Theory, $N^1(X) = H^2(X, \mathbb{R}) \cap H^{1,1}$.

We denote by $NE(X) \subset N_1(X)$ be the cone of effective 1-cycles, that is

$$NE(X) = \{C \in N_1(X) : C = \sum r_i C_i r_i \in \mathbb{R}, r_i \geq 0\},$$

where C_i are irreducible curves. Let $\overline{NE(X)}$ be the closure of $NE(X)$ in the real topology of $N_1(X)$. This is called the *Kleiman–Mori cone*.

We will also use the following notation:

$$\overline{NE(X)}_{H \geq 0} := \overline{NE(X)} \cap H_{\geq 0} \text{ and similarly for } > 0, \leq 0, < 0.$$

One effect of taking the closure is the following trivial observation, which has many important use in applications: if $H \in N^1(X)$ is positive on $\overline{NE(X)} \setminus 0$ then the section $(Hz = 1) \cap \overline{NE(X)}$ is compact. Indeed, the projectivised of the closed cone $\overline{NE(X)}$ is a closed subset of $\mathbb{P}^{\rho-1} = \mathbb{P}(N_1(X))$, and therefore compact, and the section $(Hz = 1)$ projects homeomorphically to it. The same holds for any face or closed sub-cone of $\overline{NE(X)}$.

An element $H \in N^1(X)$ is called *numerically eventually free* or *nef* if $HC \geq 0$ for every curve $C \subset X$ (in other words if $H \geq 0$ on $\overline{NE(X)}$).

The relation between nef and ample divisors is the content of the following Kleiman criterion that is a basic stone of the Mori theory.

Theorem 1.1.1 ([Kle]). *For $H \in Pic(X)$, view the class of H in $N^1(X)$ as a linear form on $N_1(X)$. Then*

$$H \text{ is ample} \iff HC > 0 \text{ for all } C \in \overline{NE(X)} \setminus \{0\}.$$

In other words the theorem says that the cone of ample divisors is the interior of the nef cone in $N^1(X)$.

Note that it is not true that $HC > 0$ for every curve $C \subset X$ implies that H is ample. The condition in the theorem is stronger.

This is only a weak form of Kleiman's criterion, since X is a priori assumed to be projective. The full strength of Kleiman's criterion gives a necessary and sufficient condition for ampleness in terms of the geometry of $\overline{NE(X)}$.

The first main theorem of Mori theory is the following description of the negative part (with respect to K_X) of the Kleiman-Mori cone.

Assume that X is smooth and denote by K_X the *canonical divisor* of X, that is an element of $Div(X)$ such that $\mathcal{O}_X(K_X) = \Omega^n_X$.

We recall also that, by definition, a *rational curve* is an irreducible, reduced curve defined over k whose normalization is $\mathbb{P}^1$.

Theorem 1.1.2 ([Mo3]). *Let X be a non singular projective variety.*

1) There are countably many rational curves $C_i \subset X$ such that $0 < -C_i K_X \leq dim X + 1$ and

$$\overline{NE(X)} = \overline{NE(X)}_{K_X \geq 0} + \sum \mathbb{R}_{\geq 0}[C_i].$$

2) For any $\epsilon > 0$ and ample divisor H,

$$\overline{NE(X)} = \overline{NE(X)}_{K_X+\epsilon H\geq 0} + \sum_{finite} \mathbb{R}_{\geq 0}[C_i].$$

In simple words the theorem says the following: consider the linear form on $N_1(X)$ defined by K_X. The part of the Mori-Kleiman cone $\overline{NE(X)}$ which stays in the negative semi-space defined by this linear form (if not empty) is locally polyhedral and it is spanned by a countable number of *extremal rays*. Each extremal ray is spanned in $N_1(X)$ by a rational curve with bounded intersection with the linear form. Moreover, moving an ϵ away from the hyperplane (in the negative direction) the number of extremal rays become finite.

There are essentially two ways of proving this theorem; the original one, which is due to Mori, is very geometric and valid in any characteristic. It is presented in the paper [Mo3] and in many other places, for example in [KM2]. It is based on deformations of rational curve on an algebraic variety and it makes strong use for instance of the theorem 4.11.13.

The other proof was provided by Y. Kawamata ([Ka0]); it gives the cone theorem as a consequence of the following rationality theorem.

Theorem 1.1.3 ([KMM] 4.1.1 Kawamata rationality theorem)**.**

Let X be an n- dimensional variety defined over $\mathbb{C}$ which is smooth or, more generally with LT singularities (see Definition 2.4.1), for which K_X is not nef.

Let L be an ample line bundle on X and define the nef value (or threshold) of the pair (X, L) by

$$r = \inf\{t \in \mathbb{R} : K_X + tL \text{ is nef}\}.$$

Then the nef value is a rational number.

Moreover if $a = 1$ or more generally in the singular case $a := min\{e \in \mathbb{N} : eK_X \text{ is Cartier}\}$, and if $ar = v/u$, with $(v, u) = 1$, then $v \leq a(n + 1)$.

The proof of this proposition uses the technique of the base point free theorem which we will introduce in the next section. In particular it makes use of vanishing theorems and it is therefore valid only in characteristic zero.

It was noticed by M. Reid and Y. Kawamata that the rationality theorem and the Base point free theorem implies immediately the Mori's cone theorem, in the more general case of varieties with L-T singularities.

A very nice presentation of the above theorems (Kleiman-Mori-Kawamata), together with complete proofs, in the **case of surfaces** is in [Re4], Chapter D. This material can be presented in few hours (3-4) to an audience with a limited knowledge of basic algebraic geometry and it can provide a good insight in the field; this is our experience at the Ferrara course.

The surface case is a perfect tutorial case in order to understand the Minimal Model Program. This was first pointed out by S. Mori who worked

out a complete description of extremal rays in the case of a smooth surface (see [[Mo3], chapter 2], see also [[KM2], P. 21, 22, 23 and section 1.4]). Moreover he also showed how it is possible to associate to each extremal ray a morphism from the surface. When the ray is spanned by a rational curve with self intersection -1, this is a celebrated theorem of Castelnuovo. Castelnuovo's whose proof is also very enlighting and it can be found in [[Be], théorém II.17] or in [[Ha], theorem V. 5.7].

1.2. Fujita $\Delta - genus$

A classical approach to the classification of projective varieties, which dates back to the Italian school, consists in taking an hyperplane section, characterize it inductively and subsequently describe the original variety. To stress its classical flavor T. Fujita called it *Apollonius method*; we will now introduce some definitions and techniques as presented in the work of T. Fujita ([Fu2]).

Definition 1.2.1. Let F be a variety of dimension d and let L be an ample line bundle on F; the pair (F, L) is called a a polarized variety. We will denote with

$$\chi(F, tL) = \Sigma \chi_j \frac{(t(t+1)...(t+j-1)}{j!}$$

the Hilbert polynomial of (F, L); the positive integer $\delta(F, L) := \chi_n = L^d$ is called the degree of (F, L) and $g(F, L) := 1 - \chi_{(n-1)}$ is called the sectional genus. The $\Delta - genus$ of (F, L) is defined by the formula

$$\Delta(F, L) = d + \delta - h^0(F, L).$$

Definition 1.2.2. . Let (F, L) be a polarized variety. Let D be a member of $|L|$ and suppose that D is irreducible and reduced as a subscheme of F; in such a case D is called a rung of (F, L). Let $r : H^0(F, L) \to H^0(D, L_D)$ be the restriction map; if r is surjective the rung is said to be a regular rung.

A sequence $F = F_d \supset F_{n-1} \supset .... \supset F_1$ of subvariety of F such that F_i is a rung (a regular rung) of (F_{i+1}, L_{i+1}) is called a ladder (a regular ladder).

Remark 1.2.3. The pair (D, L_D) is a polarized variety of dimension $d - 1$. The structure of (F, L) is reflected in that of (D, L_D). One can study (F, L) via (D, L_D) using induction on d. This is the main idea of the Apollonius method. In particular we have that $\chi(D, tL_D) = \chi(F, tL) - \chi(F, (t-1))L)$, $g(D, L_d) = g(F, L)$, $\delta(D, L_D) = \delta(F, L)$ and $\Delta(F, L) - \Delta(D, L_D) = dimCoker(r)$; therefore if the rung is regular the two $\Delta - genus$ are the same.

In classical geometry the number $dimCoker(r)$ was called deficiency.

Assume that L is very ample and let φ_L be the map associated to the global sections of L. Then it is a classical result that $\Delta(F, L) \geq 0$ and equality holds for the so called "Varieties of Minimal Degree". This varieties are classified as projective spaces, hyperquadrics, scroll over a rational curves or generalized cones over them.

A modern approach to this classification which extend to the case when L is merely ample is due to T. Fujita ([Fu2]).

Proposition 1.2.4. *Let (F, L) be a polarized variety and assume that there exists a ladder for this pair; then $\Delta(F, L) \geq 0$ (this is actually always true, without the assumption of the existence of a ladder). If moreover the ladder is regular and for a divisor $D_1 \in |L_{|F_1}|$ the map $H^0(F_1, L_{|F_1}) \to H^0(D, L_{|D}) = \mathbb{C}^\delta$ is surjective (we will call this a complete regular ladder) then $\Delta(F, L) = 0$ and the pair (F, L) is a variety of minimal degree; in particular F is normal, $g(F, L) = 0$ and L is very ample.*

Proof. The proof follows immediately from the above observations plus the fact that the surjectivity of $H^0(F_1, L_1) \to H^0(D, L_D) = \mathbb{C}^\delta$ implies that D_1 is a rational normal curve. □

Exercise 1.2.5. *Let F_1 be a curve and L a line bundle on F_1. Assume that $D \in |L|$ is an effective divisor such that $H^0(F_1, L_1) \to H^0(D, L_D) = \mathbb{C}^\delta$ is surjective. Prove that F_1 is a rational normal curve.*

Exercise 1.2.6. *Give a sketch of proof of the classification of Varieties of Minimal degree, see* [GH].

Part 2. Base point free technique

In this part we will introduce the Base point free technique (for short BPF). This theory has been mainly developed by Kawamata, Reid, Shokurov in a series of papers; we suggest [KMM], [Ko2] and [Ka3]. The aim of BPF is to show that an adjoint linear system, under some conditions, is free from fixed points. In the first section we will try to prevent the reader from a too technical approach, giving the main ideas and results, without too many definitions and details. The latter are left for the interested reader, together with example and exercises, in the last section.

2.3. Base point freeness

How to construct morphisms on a given projective variety is the contest of this section.

Let C be a compact Riemann surface of genus g and let K_C be the canonical bundle of C. To give a morphism $C \to \mathbb{P}^N$ is equivalent to give a line bundle H without base points. For this we have the very well known

Theorem 2.3.1. *If $degH \geq 2g$ then H has no base point.*

Proof. Let $L := H - K_C$ and let $x \in C$ be a point on C; note that by assumption $degL \geq 2$ and thus

$$H^1(X, K_C + L - x) = H^0(C, x - L) = 0, \tag{2.3.1}$$

the first equality coming from Serre duality.

Then we consider the exact sequence

$$0 \to \mathcal{O}_C(K_C + L - x) \to \mathcal{O}_C(K_C + L) \to \mathcal{O}_x(K_C + L) \to 0,$$

which comes by tensoring the structure sequence of x on C,

$$0 \to \mathcal{I}_x \to \mathcal{O}_C \to \mathcal{O}_x \to 0,$$

by the line bundle $K_C + L$.

The sequence gives rise to a long exact sequence in cohomology whose first terms are

$$0 \to H^0(C, K_C + L - x) \to H^0(C, K_C + L) \to H^0(x, K_C + L) \to 0.$$

In particular we have the surjective map

$$H^0(C, H) \xrightarrow{\alpha} H^0(x, H) \to 0.$$

Furthermore x is a closed point and therefore

$$H^0(x, H) = \mathbb{C} \neq 0. \tag{2.3.2}$$

The surjectivity of α translates into the existence of a section of $\mathcal{O}(H)$ which is not vanishing at x, it is the pull back via α of 1. □

What we have done can be summarized in the following slogan, which is somehow the manifesto of the base point free technique.

Construct section of an adjoint line bundle proving a vanishing statement, (2.3.1), and a non vanishing on a smaller dimensional variety, (2.3.2).

What happens if we try to generalize this to higher dimensional varieties and which problems will we encounter ?

Simple observation: the point $x \in C$ is a smooth Cartier divisor, that is why with an abuse of language we wrote $H^i(C, K_C + L - x)$. This is no more true for a variety X of higher dimension.

Let $x \in X$ be a point of a smooth projective variety X of dimension n. If we just want to mimic the above arguments, then in (2.3.1) we are concerned with cohomology groups of non locally free sheaves which are of difficult interpretation. Note that there is a way to make a divisor out of a point: blow it up! Do it and get a morphism $\pi : Y \to X$ with exceptional divisor E and

$$\pi^* K_X = K_Y - (n-1)E$$

Assume now that we want to prove that x is not a base point of a divisor of the type $H := K_X + L$; we pull back the divisors on Y and we have an exact sequence, coming from the structure sequence associated to E, of the type

$$H^0(Y, \pi^*(K_X + L)) \to H^0(E, \pi^*(K_X + L)) \to H^1(Y, \pi^*(K_X + L) - E).$$

Since $H^0(Y, \pi^*(K_X + H)) = H^0(X, K_X + L)$ (Hartogs theorem) and $H^0(E, \pi^*(K_X + L)) = \mathbb{C}$, we have to prove "only" the vanishing of

$$H^1(Y, \pi^*(K_X + L) - E) = H^1(Y, K_Y + \pi^* L - nE).$$

This is of course in general not true and one has to choose carefully good assumptions on L to have a vanishing theorem of this type; let us state the best available version of it (apart generalizing it to a relative or to a singular situation) which is due to Kawamata-Viehweg (they worked on previous versions of Enriques-Kodaira-Ramanujam-...)

Theorem 2.3.2 (Vanishing Theorem, see [KMM] or [EV]).

Let X be a smooth variety and let $D = \sum a_i D_i$ be a $\mathbb{Q}$-Cartier divisor satisfying the following conditions:
i) D is nef and big.
ii) $\langle D \rangle$ has support with only normal crossing (that is each D_i is smooth and they intersect everywhere transversally).

Then

$$H^j(X, K_X + \lceil D \rceil) = 0 \text{ for } j > 0.$$

Let us show how to use this vanishing theorem under **very special hypothesis**.
Assume *that L is ample (or nef and big) and that we can find a divisor $D_1 \in |L|$ such that*

$$\pi^*(D_1) = (\tilde{D_1}) + aE$$

with $c := \frac{n}{a} < 1$ and $(\tilde{D_1})$ smooth. Then x is not a base point of $K_X + L$

Proof Note that for every $0 < \delta << 1$ we have that $\pi^*(L) - \delta E := A$ is ample. Then we can write

$$\pi^*(K_X + L) - E \equiv K_Y + c(\tilde{D_1}) + caE - nE + (1-c)A + (1-c)\delta E$$

or

$$\pi^*(K_X + L) - E - (1-c)\delta E - c(\tilde{D_1}) - K_Y$$

is an ample $\mathbb{Q}$-divisor.

We can apply the vanishing theorem and conclude that

$$H^1(Y, \pi^*(K_X + L) - E) = 0$$

since the round up of $-(1-c)\delta E - c(\tilde{D_1})$ is zero.

Unfortunately it is very unlikely that after a simple blow-up one have the assumption satisfied. Now it comes the moment to give a precise general statement and to outline its proof.

Theorem 2.3.3 (Base point freeness, [Ka0] or [KMM]). *Let X be a variety which has at most "good singularities" (i.e. smooth or LT singularities, see the definition 2.4.1) and H a Cartier divisor. Assume that H is nef and $aH - K_X =: L$ is ample for some $a \in \mathbb{N}$. Then for $m \gg 0$ the line bundle mH is generated by global sections, i.e. there exists a regular map $\varphi : X \to W$ which is given by elements in $H^0(X, mH)$ for any $m \geq m_0$.*

Remark 2.3.4. The above theorem was proved by Y. Kawamata and VV. Shokurov with a method which builts up from the classical methods of the italians and which was developed in the case of surfaces by Kodaira-Ramanujam- Bombieri.

A very significant step in the understanding of the technique was given in a beautiful paper of M. Reid (see [Re1]). More recently important improvements were achieved by Kawamata-Shokurov-Kollár-Ein and Lazarsfeld.

The big drawback is that the method as it stands is not very effective, i.e. it does not give a good bound for m (contrary to the case of curves and surfaces). Some bound can however be achieved, namely one can show that $m_0 \leq 2(n+2)!(a+n)$ (Effective Base point freeness: see [Ko4]).

We will only outline the proof and we refer to [KMM] for many technical, and often very relevant, parts which we now state and briefly comment.

We observed that a simple blow-up cannot bring us to the "perfect" assumption; then we need the following.

Definition 2.3.5. For a pair (X, H) of a variety X and a $\mathbb{Q}$-divisor H, a log resolution is a proper birational morphism $\mu : Y \to X$ from a smooth variety Y such that the union of the support of $\mu_*^{-1}H$ and of the exceptional locus is a normal crossing divisor.

Theorem 2.3.6. *Let X be a variety with LT singularities, B an effective and nef $\mathbb{Q}$ divisor and L an ample divisor on X. Then there exists a log resolution $\varphi : Y \to X$ such that*

$$K_Y = f^*K_X + \sum e_i E_i$$
$$f^*(B) = B' + \sum b_i E_i$$
$$f^*(L) = A + \sum p_i E_i$$

where all relevant divisors in Y are smooth and normal crossing, all E_i are exceptional, A is an f-ample $\mathbb{Q}$-divisor, $0 \leq p_i \ll 1$ and $e_i > -1$.

The theorem follows essentially from the work of Hironaka on resolution of singularities. The statement on the e_i is the definition of LT singularities (see the definition 2.4.1) while the ampleness of A is usually called Kodaira's lemma.

Using a log-resolution instead of the blow-up we will achieve our assumption but we will very likely loose the non vanishing part (namely $H^0(E, \pi^*(K_X + L)) = \mathbb{C}$). For this we need the next very important result, due to V.V. Shokurov.

Theorem 2.3.7. Non vanishing theorem *[*[KMM]*, 2.1.1] Let X be a non singular projective variety; let N be a Cartier divisor and A a $\mathbb{Q}$-divisor on X such that*
i) N is nef
ii)$\lceil A \rceil \geq 0$ and $\langle A \rangle$ has support with only normal crossing
iii) $dN + A - K_X = M$ where M is nef and big, for some positive $d \in N$.
Then $H^0(X, mN + \lceil A \rceil) \neq \emptyset$ for all $m >> 0$.

Sketch of the proof of 2.3.3.
By the non vanishing theorem there exists an effective divisor $B \in |mH|$ for all $m \geq m - 0 >> 0$.
Noetherian argument: Let $B(\gamma) = \{$ the reduced base locus of $|\gamma H|\}$. Clearly $B(\gamma^s) \subseteq B(\gamma^t)$ for any positive integers $s > t$. Noetherian induction implies that the sequence $B(\gamma^n)$ stabilizes and we call the limit B_γ. So either B_γ is non empty for some γ or B_γ and $B_{\gamma'}$ are empty for two relative prime integers γ and γ'. In the latter case, take s, t such that $B(\gamma^s)$ and $B((\gamma')^t)$ are empty and use the fact that every sufficiently large integer is a linear combination of γ^s and $(\gamma')^t$ with non negative coefficients to conclude that $|mH|$ is base point free for all $m >> 0$. So we must show that the assumption that some B_γ is non empty leads to a contradiction. Let $m = \gamma^s$ such that $B_\gamma = B(m)$ and assume that this is not empty.

With L as in the statement of the theorem and B as at the beginning of the proof, let e_i, b_i, p_i as in the theorem 2.3.6 and define

$$c := \min\{\frac{e_i + 1 - p_i}{b_i}\}$$

By taking m big enough we can assume that $0 < c < 1$; by changing the coefficients p_i a little we can assume that the minimum is achieved for exactly one index. Let us denote the corresponding divisor by E_0 and let $Z = f(E_0)$; by Bertini theorem we can assume that Z is contained in the base locus of mH, i.e. in B_c. Let $m \geq m_0$; we can write

$$\begin{aligned} &K_Y + A + cB' + \sum(cb_i - e_i + p_i)E_i + f^*(m - cm)H \\ &\equiv f^*(K_X + L + mH) = f^*(m + a)H = f^*(N(m)) \end{aligned}$$

and

$$\sum(cb_i - e_i + p_i)E_i = E_0 - D + Fr$$

where E_0, D are effective divisor without common irreducible components and Fr is the fractional divisor with rational coefficients between 0 and 1, defined by $Fr = \sum\{cb_i - e_i + p_i\}E_i$.

Thus

$$f^*((m + a)H) + D - E_0 - Fr - cB' - K_Y$$

is ample, thus, denoting $N(m) := f^*((m + a)H) + D$ for brevity, by the vanishing theorem we have then

$$H^i(Y, N(m) - E_0) = 0 \text{ for } i > 0$$

and

$$H^i(E_0, N(m)) = 0 \text{ for } i > 0$$

By the first vanishing the restriction map

$$H^0(Y, N(m)) \to H^0(E_0, N(m)_{|E_0})$$

is surjective.

By the non vanishing theorem for any $m_1 \geq m_0$ and a non-zero section s of $N(m_1)_{|E_0}$. By surjectivity this extends to a non-zero section of $N(m_1)$

on Y; which is not identically zero along E_0. Moreover $H^0(Y, N(m_1)) = H^0(X, (m_1 + a)H)$ since D is f-exceptional. The section s descends to a section of $(m_1 + a)H$ which does not vanish along $f(E_0) = Z \subset B_c$, which is a contradiction.

2.4. Singularities and log singularities.

In the previous sections we did not introduce any technical definitions of singularities or of singular pairs. Let us do it now for the interested reader.

Definition 2.4.1. Let X be a normal variety and $D = \sum_i d_i D_i$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. If $\mu : Y \to X$ is a log resolution of the pair (X, D), then we can write

$$K_Y + \mu_*^{-1} D = \mu^*(K_X + D) + F$$

with $F = \sum_j disc(X, E_j, D) E_j$ for the exceptional divisors E_j. We call $e_j := disc(X, E_j, D) \in \mathbb{Q}$ the discrepancy coefficient for E_j, and regard $-d_i$ as the discrepancy coefficient for D_i.

The variety X is said to have terminal (respectively canonical, log terminal (LT)) singularities if $e_j > 0$ (resp. $e_j \geq 0$, $e_j > -1$), for any j.

The pair (X, D) is said to have log canonical (LC) (respectively Kawamata log terminal (KLT)) singularities if $d_i \leq 1$ (resp. $d_i < 1$) and $e_j \geq -1$ (resp. $e_j > -1$) for any i, j of a log resolution $\mu : Y \to X$.

The log canonical threshold of a pair (X, D) is $lct(X, D) := sup\{t \in \mathbb{Q}: (X, tD) \text{ is LC}\}$.

Definition 2.4.2 ([Ka3]). Let X be a normal variety and $D = \sum d_i D_i$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. A subvariety W of X is said to be a center of log canonical singularities for the pair (X, D), if there is a birational morphism from a normal variety $\mu : Y \to X$ and a prime divisor E on Y with the discrepancy coefficient $e \leq -1$ and such that $\mu(E) = W$. For another such $\mu' : Y' \to X$, if the strict transform E' of E exists on Y', then we have the same discrepancy coefficient for E'. The divisor E' is considered to be equivalent to E, and the equivalence class of these prime divisors is called a place of log canonical singularities for (X, D). The set of all centers (respectively places) of LC singularities is denoted by $CLC(X, D)$ (resp. $PLC(X, D)$), the locus of all centers of LC singularities is denoted by $LLC(X, D)$.

The study of these objects has been developed by Kawamata and we can summarize the main results in the following Theorem.

Theorem 2.4.3 ([Ka3],[Ka4]). *Let X be a normal variety and D an effective $\mathbb{Q}$-Cartier divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. Assume that X is LT and (X, D) is LC.*

i) *If $W_1, W_2 \in CLC(X, D)$ and W is an irreducible component of $W_1 \cap W_2$, then $W \in CLC(X, D)$. In particular, there exist minimal elements in $CLC(X, D)$ with respect to inclusion.*

ii) *If $W \in CLC(X, D)$ is a minimal center then W is normal*
iii) *(subadjunction formula)Let H be an ample Cartier divisor and ϵ a positive rational number. If W is a minimal center for $CLC(X, D)$ then there exist effective $\mathbb{Q}$-divisors D_W on W such that $(K_X + D + \epsilon H)_{|W} \equiv K_W + D_W$ and (W, D_W) is KLT.*

Remark 2.4.4. The first two statements are, essentially, a direct consequence of Shokurov Connectedness Lemma, which is itself a direct consequence of the Vanishing theorem 2.3.2. The subadjunction formula is quite of a different flavor and is related to semipositivity results for the relative dualising sheaf of a morphism.

In particular 2.4.3 tell us that the variety W is not too bad and there is some hope to be able to work on it.

Exercise 2.4.5. *Work out all possible minimal centers on a smooth surface, there are really not too many, and on a smooth threefold (keep in mind that KLT singularities are rational).*

2.4.1. **How to use singularities and the CLC locus to prove base point free-type theorems.** Let (X, D) a log variety and assume that (X, D) is LC and $W \in CLC(X, D)$ is a minimal center. Then we will have a log resolution $\mu : Y \to X$ with

$$K_Y = \mu^*(K_X + D) + \sum e_i E_i,$$

this time we put also the strict transform of the boundary on the right hand side. Since (X, D) is LC and $W \in CLC(X, D)$ then $e_i \geq -1$ and there is at least one $e_j = -1$ such that $\mu(E_j) = W$.

The main problem here is that to apply Kawamata's BPF method we need to have one and only one exceptional divisor with discrepancy -1 and W as a center. To fulfill this requirement we need a
Perturbation argument: Choose a generic very ample M such that $W \subset Supp(M)$ and no other $Z \in CLC(X, D) \setminus \{W\}$ is contained in $Supp(M)$, this is always possible since W is minimal in a dimensional sense. We then perturb D to a divisor $D_1 := (1 - \epsilon_1)D + \epsilon_2 M$, with $0 < \epsilon_i \ll 1$ in such a way that

- (X, D_1) is LC
- $CLC(X, D_1) = W$
- $\mu^* \epsilon_2 M = \sum m_i E_i + P$, with P ample; this is possible by Kodaira Lemma.

After this perturbation the log resolution looks like the following

$$K_Y + \sum_{j=0} E_j + \Delta - A = \mu^*(K_X + D_1) - P,$$

where the E_j's are integral irreducible divisors and $\mu(E_j) = W$, A is a μ-exceptional integral divisor and $\lfloor \Delta \rfloor = 0$. It is now enough to use the ampleness of P to choose just one of the E_j. Indeed for small enough

$\delta_j > 0$ $P' := P - \sum_{j=1} \delta_j E_j$ is still ample therefore we produce the desired resolution

$$K_Y + E_0 + \Delta' - A = \mu^*(K_X + D') - P', \tag{2.4.1}$$

here and all trough the paper after a perturbation we will always gather together all the fractional part with negative log discrepancy in P and Δ, respectively the ample part of it and the remaining.
(If instead of an ample M we choose a nef and big divisor, we can repeat the above argument with Kodaira Lemma, but this time we cannot choose the center $\mu(E_0)$ like before, and in particular we cannot assume that at the end we are on a minimal center for (X, D).)

Assume now that X is a variety with log terminal and Gorenstein singularities and let L be an ample line bundle on X; let D be an effective $\mathbb{Q}$-Cartier divisor such that $D \sim_{\mathbb{Q}} tL$ for a rational number $t < 1$. Let W be a minimal center. Perturb D using the very ample line bundle $M := mL$ for $m >>> 0$. So that we can assume that there exists only one exceptional divisor with discrepancy -1 and W as a center. Thus taking an embedded log resolution of the pair (X, D), $\mu : Y \to X$ we have

$$K_Y + E + F = \mu^*(K_X + D)$$

where E is a reduced divisor such that $\mu(E) = W$ and $F = \sum f_i F_i$ with $f_i < 1$ if $W \subset \mu(F_i)$. Then

$$K_Y + (1 - t)\mu^* L \sim_{\mathbb{Q}} \mu^*(K_X + L) - E - F$$

and thus

$$H^1(Y, \mu^*(K_X + L) - E - F) = 0$$

and we obtain a surjection

$$H^0(Y, \mu^*(K_X + L) - \lceil F \rceil) \to H^0(E, \mu^*(K_X + L) - \lceil F \rceil).$$

Since $\lceil F \rceil$ is effective and exceptional over a neighborhood of W we have that also

$$H^0(Y, \mu^*(K_X + L)) \to H^0(E, \mu^*(K_X + L) - \lceil F \rceil)$$

is surjective.

Thus to find a section of $K_X + L$ not vanishing on W it is sufficient to find a non zero section in $H^0(E, \mu^*(K_X + L) - \lceil F \rceil)$.

The **ideal case** happens when $W = x$ is one point; in this case then $K+L$ is base point free at x (since in this case $H^0(E, \mu^*(K_X + L) - \lceil F \rceil) = \mathbb{C}$)

2.5. Exercises-Examples

Exercise 2.5.1. *Let L be a very ample line bundle on a smooth projective variety X of dimension n.*
a) Prove that $K_X + (n + 1)L$ is spanned.
b) Prove that $K_X + nL$ is spanned unless $X = \mathbb{P}^n$ and $L = \mathcal{O}(1)$.

c) Prove that $K_X + (n-1)L$ *is spanned unless* $X = \mathbb{P}^n$ *and* $L = \mathcal{O}(1)$ *or* $X = \mathbb{P}^2$ *and* $L = \mathcal{O}(2)$ *or* (X, L) *is a scroll over a curve.*
d) Conjecture: $K_X + (n-2)L$ *is spanned if and only if it is nef.*

hints For the question a) let $x \in X$ and take n-sections of L meeting transversally in x

For b) use an "induction procedure"; namely take a smooth section $D \in |L|$ passing through x (this is Bertini theorem) and use the exact sequence

$$H^0(X, K_X + nL) \to H^0(D, K_D + L) \to 0.$$

One goes down until the dimension of D is 1, i.e. a curve, and in this case $K_D + L$ is spanned if and only if $degL \geq 2$. The only problem is then when D is a line and therefore $X = \mathbb{P}^n$ and $L = \mathcal{O}(1)$.

For c), one can reduce the problem to the surface case, namely $X = S$ is a smooth surface and we want to prove the spannedness of $K + L$. In this case there are even stronger theorems (Reider type theorems)

The above exercise is extremely hard when one assumes only ampleness (and not very ampleness!) of L. In fact we have:

Conjecture 2.5.2 (Fujita conjecture). Let L be an ample line bundle on a smooth projective variety of dimension n. Then $K_X + mL$ is base point free if $m \geq n + 1$ and it is very ample if $m \geq n + 2$.

Remark 2.5.3. Some very important results toward a proof of the conjecture were found in recent time. In particular, using an analytic approach, Demailly, Angern-Siu and Tsuji proved that if $m \geq \left(\frac{n+1}{2}\right)$ then $K_X + mL$ is base point free and if $m \geq \left(\frac{n+2}{2}\right)$ then the global sections of $K_X + mL$ separate points.

The base point free part of the conjecture is true in the case $n = (1), 2, 3, 4$ by results of Reider, Ein-Lazarsfeld, Kawamata and Fujita.

Part 3. Fano-Mori or extremal contractions

In this Part we will first define and give examples of Fano–Mori spaces. These are exactly the morphisms constructed in Part 2, and they play a central role in the Minimal Model Program. To study those objects we want to apply an inductive method as in section 1.2. A fundamental step is therefore to ensure that we have base point free linear systems to slice the fibers. This is the content of the Theorem 3.8.1, whose proof occupies the last section.

3.6. Contractions associated to a ray of the Kleiman–Mori cone

A key step in the Mori theory, after the description of the structure of $\overline{NE(X)}$ outlined in a previous section, is the fact that extremal rays (and in general extremal faces) correspond to morphisms of the variety. This will be explained in this section.

Proposition 3.6.1. *Let R be an extremal ray of the Mori-Kleiman cone $\overline{NE(X)}$ such that $R \cdot K_X < 0$. Then there exists a nef Cartier divisor H_R such that $H_R \cdot z = 0$ if and only if $z \in R$.*

This proposition is proved for instance in [Ko3, III.1.4.1]; the proof makes use of the Cone theorem and some easy properties of closed cones.

Then to a divisor as in the proposition we can associate a morphism via the following theorem.

Theorem 3.6.2 (Contraction theorem)**.** *Let X be a variety with log-terminal singularities and let H be a nef Cartier divisor on X.*

Assume that $F := H^{\perp} \cap \overline{NE(X)} \setminus \{0\}$ is contained in $\{C \in N_1(X) : K_X \cdot C < 0\}$; then there exists a projective morphism $\varphi : X \to W$ onto a normal projective variety W which is characterized by the following properties

i) *For any irreducible curve C on X $\varphi(C)$ is a point if and only if $H.C = 0$.*
ii) *φ has only connected fibers*
iii) *$H = \varphi^*(A)$ for some ample Cartier divisor on W.*

Proof. The proof follows immediately from the theorem 2.3.3 and Zariski's Main theorem once we note that by our assumption and Kleiman's criterion for ampleness there exists a natural number a such that $aH - K_X$ is ample. □

Definition 3.6.3. A *contraction* is a surjective morphism $f : Y \to T$, with connected fibers, between normal varieties.

For a contraction $f : Y \to T$ the set

$$E = \{y \in Y; f \text{ is not an isomorphism at } y\}$$

is the exceptional locus of f. Let $\delta = dim E$ where *dim* denotes as usual the maximum of the dimension of the irreducible components.
f is called of *fiber type* if $\delta = dim Y$ or *birational* otherwise.

If f is birational and $\delta = dim Y - 1$ then it is also called a *divisorial contraction*; if it is birational and $\delta \leq dim Y - 2$ then it is called a *small contraction*.

For a contraction $f : Y \to T$ a Cartier divisor H such that $H = \varphi^*(A)$ for some ample Cartier divisor on T is called a supporting divisor for the contraction (if $H = H_R$ as in the above proposition then it is also called a supporting divisor for the ray R).

Definition 3.6.4. A contraction $f : X \to W$ as in the above Theorem 3.6.2 is called *Fano–Mori* (F-M) or *extremal*.

A birational contraction $f : X \to W$ is called *crepant* if $K_X = f^* K_W$

Remark 3.6.5. Putting together Theorem 3.6.2 and Proposition 3.6.1 we obtain the following. Given an extremal ray of the Mori-Kleiman cone $R \subset \overline{NE(X)}$ such that $R \cdot K_X < 0$, there exists a projective morphism with

connected fibers $cont_R : X \to W$ onto a normal projective variety W, which contracts all (and only) the curve in the ray.

Such a map is also called *the contraction of the extremal ray* R, or an *elementary Fano-Mori contraction.*

We stress that this is proved only in the case $char(k) = 0$. Indeed Theorem 3.6.2 is proved only in characteristic zero. The existence of this map is an open problem in positive characteristic (if $dim X \geq 4$).

Remark 3.6.6. It is obvious that vice-versa the Contraction theorem implies the Theorem 2.3.3.

Note also that any supporting divisor H for a F-M contraction φ is of the type $K_X + rL$ with r a rational number and L an ample Cartier divisor. In fact let H be a Cartier divisor which is the pull back of a sufficiently ample line bundle on W; then $m(H - K_X) := L$ is an ample Cartier divisor for some rational number m and thus $H = K_X + 1/mL$.

Remark 3.6.7. To construct a divisor as in 3.6.2, and therefore an associated morphism, one can also use the Rationality theorem 1.1.3 as follows. Let X be a variety with at most log-terminal singularities and let L be a Cartier divisor with nef value r. Then, by the rationality theorem, if $H' := K + rL$ there exists an integer m such that $H := mH'$ is a Cartier divisor. By definition H satisfies the assumption in 3.6.2.

The following is an important technical result whose proof is an interesting exercise .

Exercise 3.6.8. *(*[KMM, Proposition 5.1.6]*) Let $f : X \to W$ be a divisorial elementary Fano Mori contraction with X smooth or with at most terminal $\mathbb{Q}$-factorial singularities. Then the exceptional locus of f is a unique prime divisor and W has at most terminal $\mathbb{Q}$-factorial singularities.*

3.6.1. **Local Contraction.** To understand locally a F-M contraction one has to restrict to an affine neighborhood of a fixed fiber; then one can "glue" different local descriptions and obtain more general complete F-M contractions.

For this we use the local set-up developed by Andreatta–Wiśniewski, see [AW1], which depends on some definitions.

Definition 3.6.9. Let $f : Y \to T$ a contraction supported by $K_Y + rL$, with r rational and L ample and Cartier. Fix a fiber F of f and an open affine $S \subset T$ such that $f(F) \in S$. Let $X = f^{-1}S$ then $f : X \to S$ will be called a local contraction around F_0. If there is no need to specify fixed fibers then we will simply say that $f : X \to S$ is a local contraction. In particular $S = Spec(H^0(X, \mathcal{O}_X))$.

Definition 3.6.10. Let $f : X \to S$ a local F-M contraction around F. Assume that f is supported by $K_Y + rL$. The Cartier divisor L will be called fundamental divisor of f. Let G a generic non trivial fiber of f then the dual-index of f is

$$d(f) := dim G - r,$$

the character of f is

$$\gamma(f) := \begin{cases} 1 & \text{if } dimX > dimS \\ 0 & \text{if } dimX = dimS \end{cases}$$

and the difficulty of f is

$$\Phi(f) = dimF - r.$$

We will say that $(d(f), \gamma(f), \Phi(f))$ is the type of f.

3.7. Examples

.

A large class of examples of F-M contractions is worked out in section 3 of the paper [AW3]; we report some of them here, referring the reader for more details to that paper.

We focus on the case X is smooth, with the purpose of showing later some classifications of a smooth variety.

Example 3.7.1. Fano varieties, Scrolls (i.e. $X = \mathbb{P}(E)$ where E is a vector bundle on Y), conic bundles, are F-M contractions of fiber type.

Example 3.7.2. Any smooth blow up on a smooth variety is a birational F-M contraction.

Blow-up a smooth surface in a 4-fold with an ordinary double point; i.e.

$$S := \{x = z = w = 0\} \subset Z := \{xy - zt + w^2\}$$

$$\varphi : X := Bl_S Z \to Z.$$

A direct computation shows that X is smooth and that $\varphi^{-1}(0) = \mathbb{P}^2$.

Example 3.7.3. Let L_1, L_2, L_3 be three general plane in $\mathbb{P}^3$ and let $\mathbb{P}^2$ be the base of the net $\mathcal{L} = \Sigma t_i L_i$. Consider the incidence variety

$$X := \{(p, L); p \in L\} \subset \mathbb{P}^3 \times \mathbb{P}^2.$$

Then the projection $\varphi : X \to \mathbb{P}^3$ is a F-M contraction which is a $\mathbb{P}^1$-bundle generically and has a fiber $= \mathbb{P}^2$ over the point intersection of the L_i.

If we blow-up a smooth surface S in X meeting the general fiber in one point we obtain a smooth conic bundle $Y \to \mathbb{P}^3$ with a two dimensional reducible fiber and with discriminant locus $\Delta = \varphi(S)$. On Y there is another F-M contraction, of birational type, The one of example 3.7.2.

Example 3.7.4. The existence of F-M birational contractions with exceptional set of codimension greater than 1 (small contraction) was proved by P. Francia with a famous example: it is a F-M contraction on a 3-fold with terminal singularities and with exceptional locus $E \cong \mathbb{P}^1$. The example is worked out in many books, for instance in ([CKM], p.33-34). This is a main difficulty in the MMP over-passed by S. Mori, with a tremendous work, in dimension 3 (see [Mo4]).

Example 3.7.5. We now introduce a large class of examples via a standard construction; for more details see section 3 of [AW3].

Let $\mathcal{E}$ be a vector bundle over a smooth variety F and let $\mathbf{V}(\mathcal{E}) := Spec(S(\mathcal{E}))$ be the *total space of the dual* $\mathcal{E}^*$.

If $S^k(\mathcal{E})$ is *generated by global sections* for some $k > 0$ let

$$\varphi : \mathbf{V}(\mathcal{E}) \to Z = Spec(\bigoplus_{k\geq 0} H^0(F, S^k(\mathcal{E})),$$

be the map associated to the evaluation of $S^k(\mathcal{E})$. φ is a contraction. The collapsing of the zero section , $F_0 := F$, of the total space $\mathbf{V}(\mathcal{E})$ to the vertex z of the cone Z.

It is straightforward to check the following properties:

i) The normal bundle of F_0 into $\mathbf{V}(\mathcal{E})$ is $\mathcal{E}^*$.

ii) If $-K_Y - det\mathcal{E}$ is ample then φ is a Fano-Mori contraction. The map is birational if the top Segre class of $\mathcal{E}$ is positive (if $rank\mathcal{E} = 2$ then $c_1^2 - c_2 > 0$).

iii) $\mathbb{P}(\mathcal{O} \oplus \mathcal{E}) := Proj(S(\mathcal{E} \oplus \mathcal{O}_Y))$ is the projective closure of $\mathbf{V}(\mathcal{E})$. The map φ is the restriction of the map given by the tautological bundle ξ on $Proj(S(\mathcal{E} \oplus \mathcal{O}_Y))$; φ is birational if ξ is big.

iv) (Grauert criterium). $\mathcal{E}$ is ample if and only if the map φ is an isomorphism outside F_0.

v) The fiber F_0 of the map φ has the fiber structure (i.e. $\mathcal{I}_{F_0} = \varphi^{(-1)} m_z \mathcal{O}_X$) if and only if $\mathcal{E}$ is spanned by global sections.

Let us work out in details the example with $F = \mathbb{P}^2$; it is possible to do the same for a two dimensional quadric (even singular or reducible), see [AW3], or for smooth del Pezzo surfaces.

Let $\mathcal{E}$ be a rank-2 vector bundle over $\mathbb{P}^2$ such that $\mathcal{E}$ is spanned by global sections and $0 \leq c_1(\mathcal{E}) \leq 2$. These bundles were completely classified in [SW], and they are isomorphic to one of the bundles in the following table. Performing the above construction with them we obtain eight Fano-Mori contractions with fiber $\mathbb{P}^2$;in the second column we describe these contractions. In [AW3] it is explained how to obtain these descriptions, one has to use the results 4.13.1, 4.13.3, 4.13.2 in the next.

description of bundle $\mathcal{E}$	description of φ and $Sing(Z)$
$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}$	a scroll, is smooth
$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(1)$	a smooth blow-up of a smooth curve, smooth
$\mathcal{E} = T\mathbb{P}^2(-1)$	a generalized scroll,i.e. a one dimensional scroll with a two dimensional fiber, smooth
$\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(2)$	the blow-up of a smooth curve which is $= Sing(Z)$
$\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(1)$	the small contraction, Z is singular and the flip exists
$0 \to \mathcal{O} \to T\mathbb{P}^2(-1) \oplus \mathcal{O}(1) \to \mathcal{E} \to 0$	the blow-up of a smooth surface passing from a quadric singularity of Z
$0 \to \mathcal{O}(-1)^{\oplus 2} \to \mathcal{O}^{\oplus 4} \to \mathcal{E} \to 0$	the blow-up of a surface with a singularity of type: cone over a twisted cubic in a smooth Z
$0 \to \mathcal{O}(-2) \to \mathcal{O}^{\oplus 3} \to \mathcal{E} \to 0$	a conic bundle with a two dimensional fiber, Z is singular

3.8. Relative Base point freeness on Fano-Mori Contractions

A F-M contraction has a supporting divisor of the type $K_X + rL$ with L an ample Cartier divisor,as noticed in 3.6.6.

This set-up, which we can also call *adjoint contraction morphism*, allows us to apply an inductive method which is typical of this theory. It was called *"Apollonius method"* by Fujita in [Fu2] while in [AW1] it is called *horizontal slicing* argument (sometimes it is called simply *slicing* but we will need to distinguish it from *vertical slicing*). It can be briefly summarized as follows.

Consider a general divisor X' from the linear system $|L|$ (a hyperplane section of X if L is very ample) and assume that it is a "good" variety of dimension $n-1$ (i.e. has the same singularities as X). By adjunction, $K_{X'} = (K_X + L)_{|X'}$ and by Kodaira-Kawamata- Viehweg Vanishing Theorem, if $r > 1$, the linear system $|m(K_{X'} + (r-1)L)|$ is just the restriction of $|m(K_X + rL)|$, so that the *adjoint contraction morphism* of X' can be related to the one of X. Moreover, fibers of the *adjoint morphism* of X' will be usually of smaller dimension and an inductive argument can be applied. For an example of the use of this method see 4.11.2.

The *horizontal slicing* argument requires therefore the existence of a "good" divisor X' in the linear system $|L|$ (a *rung* in the language of [Fu2], see section 1.2). The system, however, for an ample (but not very ample) L may a priori be even empty. To overcome this difficulty we use the local set-up, described in the previous section, in which the base of the *contraction morphism* will be affine. We will benefit from this situation also because we will be able to choose effective divisors which are rationally trivial. Then

the next point is to ensure that the divisor X' does not contain the whole fiber in question and has good singularities. This is the case for instance, via Bertini theorem, if we ensure that base point locus of $|L|$ (L may be changed by adding a divisor trivial on fibers of φ) is empty. This is what may be called a "relative good divisor". (Now we can explain why we use the word "horizontal": we are used to think about the map $\varphi : X \to Z$ as going vertically, then every divisor from an ample linear system cuts every "vertical" fiber of φ of dimension ≥ 1, so it lies "horizontally".)

The above point of view was first exploited in [AW1] where the first part of the following theorem was proved. The proof used the Base Point Free Theorem -method (Bpf-method) of Y. Kawamata, actually a slightly improved version of it by J. Kollár, see [Ko2], introduced in the previous section. The further refinement of the method by Y. Kawamata in [Ka3] and [Ka4] allowed an improvement of the theorem in [AW1], this is the second part of the following theorem and it was proved in [Me3].

Theorem 3.8.1. *Let $f : X \to S$ be a local F-M space around F supported by $K_X + rL$ and let $(d(f), \gamma(f), \Phi(f))$ be the type of f, see 3.6.9 and 3.6.10. Assume one of the two following conditions is satisfied*

- *$dim F < r+1$ or, if f is birational, $dim F \leq r+1$; equivalently the type of f is $(*, *, \Phi(f))$, with $\Phi(f) \leq 1 - \epsilon\gamma(f)$ (see* [AW1]*),*
- *the type of f is $(d, 1, 1)$, with $d \leq 0$ or with $d = 1$ and F is reducible (see* [Me3]*).*

Then L is relatively spanned, i.e $Bsl|L| := Supp(Coker(f^*f_*L \to L))$ does not meet F.

In the next part of this section we are going to prove the above theorem. Let us first roughly summarize the general principles in the proof. We want to show that the fundamental divisor is relatively free. The idea is to proceed by contradiction. Assume there is a Base locus V. Then produce a log variety non KLT on V (with respect to a divisor in δL). Then use the method developed in Part 2 to produce sections of an adjoint line bundle non vanishing along the non KLT part of log varieties. To apply this strategy we have a priori a main problem: namely BPF produces sections of $K_X + mL$, for $m \gg 0$, while we need sections of L itself. But in the category of local F–M contractions we have that $L \equiv_f K_X + (r+1)L$. An immediate consequence is the following.

Crucial observation 3.8.2. In the F-M contraction case we are working with log pairs (X, D) such that $D \equiv_f \delta L$ and $K_X \equiv_f -rL$, therefore by subadjunction formula

$$K_W + D_W \equiv_f (\delta - r + \epsilon)L.$$

In particular if W is contained in a fiber and $\delta < r$ then $K_W + D_W$ is antiample.

Definition 3.8.3. A log-Fano variety is a KLT pair (X, Δ) such that for some positive integer m, $-m(K_X + \Delta)$ is an ample Cartier divisor. The index of a log-Fano variety $i(X, \Delta) := sup\{t \in \mathbb{Q} : -(K_X + \Delta) \equiv tH$ for some ample Cartier divisor $H\}$ and the H satisfying $-(K_X + \Delta) \equiv i(X, \Delta)H$ is called fundamental divisor.

From our point of view these varieties are extremely important because we have a simple effective non vanishing, directly coming from Hilbert polynomial.

Proposition 3.8.4 ([Al],[Am]). *Let (X, Δ) be a log-Fano n-fold of index $i(X)$, H the fundamental divisor in X. If $i(X) > n - 3$ then $h^0(X, H) > 0$, moreover if $i(X) \geq n - 2$ then $h^0(X, H) > 1$.*

Proof. For simplicity assume that $\Delta = 0$ and $i(X) \geq n - 2$, the other cases are treated similarly putting some more effort. Let $p(t) := \chi(X, tH) = \sum h_j t^j$ the Hilbert polynomial of H and $d = H^n$ (see section 1.2). In particular

$$h_n = d/n!$$

and

$$h_{n-1} = \frac{-K_X \cdot H^{n-1}}{2(n-1)!} = \frac{i(X)d}{2(n-1)!}.$$

By the vanishing theorem 2.3.2

$$H^i(X, tH) = H^i(X, K_X + (tH - K_X)) = H^i(X, K_X + (i(X) + t)H) = 0$$

for $i > 0$ and $t > -i(X)$. On the other hand, H is an effective non trivial divisor therefore

$$H^0(X, tH) = 0 \quad \text{for any } t < 0.$$

Combining the two we obtain that

$$p(t) = 0 \quad \text{for } -i(X) < t < 0,$$

and $p(1) = 1$. Plug this informations into $p(t)$ to get

$$\begin{aligned} p(t) &= \frac{d}{n!}(t+1)(t+2)\dots(t+n-2)(t^2 + at + \frac{n(n-1)}{d}) \\ &= \frac{d}{n!}t^n + \frac{d}{n!}(a + \frac{(n-2)(n-1)}{2})t^{n-1} + \dots \end{aligned}$$

To determine a use

$$h_{n-1} = \frac{i(X)d}{2(n-1)!}.$$

So that

$$a = \frac{ni(X) - (n-2)(n-1)}{2}.$$

Which yields $h^0(X, H) = p(1) > d/n + (n-1) > 1$. □

Exercise 3.8.5. *If the index $i(X)$ is sufficiently close to the dimension of X then the above argument allows to characterize the variety X simply using the polynomial $p(t)$. The following characterization of Projective spaces and Quadrics, due to Kobayashi–Ochiai, can be easily deduced in this way.*

Namely one can prove that $i(X) \leq dimX+1$; moreover $i(X) = dimX+1$ if and only if $X \simeq \mathbb{P}^{dimX}$ while $i(X) \geq dimX$ if and only if either $X \simeq \mathbb{P}^{dimX}$ or $X \simeq \mathbb{Q}^{dimX}$.

We translate Proposition 3.8.4 in the non vanishing theorem we need in the following way.

Lemma 3.8.6. *Let $f : X \to S$ be a local contraction supported by $K_X + rL$ around F. Fix a subvariety $Z \subset F$, and a $\mathbb{Q}$-divisor D, with $D \equiv_f \gamma L$. Assume that X is LT, (X, D) is LC along Z, and $W \in CLC(X, D)$ is a minimal center contained in Z. Assume that one of the following conditions is satisfied:*

i) $r - \gamma > max\{0, dimW - 3\}$,
ii) $dimW \leq 1$ *and* $r - \gamma > -1$.

Then there exists a section of $|L|$ not vanishing identically on W.

Proof. Since D is LC along W we can assume, up to a perturbation, that there exists a log resolution $\mu : Y \to X$ of (X, D) with

$$K_Y - A + E + \Delta + B = \mu^*(K_X + D) - P,$$

where:

- E is an irreducible integral divisor,
- A and B are integral divisors,
- Δ and P are $\mathbb{Q}$-divisors such that,

Furthermore these divisors satisfy the following:

- $\mu(E) = W$,
- A is μ-exceptional,
- $\lfloor \Delta \rfloor = 0$,
- $Z \cap \mu(B) = \emptyset$
- P is $(f \circ \mu)$-ample.

Let

$$N(t) := \mu^* tL + A - \Delta - E - B - K_Y \equiv_{f\circ\mu} \mu^*(t + r - \gamma)L + P, \tag{3.8.1}$$

then $N(t)$ is $(f \circ \mu)$-ample whenever $t + r - \gamma \geq 0$. In particular if conditions i) or ii) of the Lemma are satisfied, by vanishing theorem 2.3.2, we have the following surjection

$$H^0(Y, \mu^* L + A - B) \to H^0(E, (\mu^* L + A)_{|E}).$$

Since A does not contain E and is effective then

$$H^0(W, L_{|W}) \hookrightarrow H^0(E, (\mu^* L + A)_{|E}).$$

In particular any section of $H^0(W, L_{|W})$ gives rise to a section in $H^0(X, L)$ not vanishing identically on W. Therefore to conclude the proof it is enough to produce a section in $H^0(W, L_{|W})$. By subadjunction formula of Theorem 2.4.3 there exists a $\mathbb{Q}$-divisor D_W such that

$$K_W + D_W \equiv (K_X + D + \epsilon L)_{|W} \equiv -(r - \gamma - \epsilon)L_{|W}, \tag{3.8.2}$$

for any $0 < \epsilon \ll 1$.

In case (i) since $r - \gamma > 0$ then by equation (3.8.2), for sufficiently small ϵ, (W, D_W) is a log Fano variety of index $i(W, D_W) = r - \gamma - \delta > dim W - 3$. Therefore we can apply Proposition 3.8.4.

If $dim W = 1$ then W is smooth. Since $r - \gamma - \epsilon > -1$ by relation (3.8.2)

$$0 < L \cdot W \geq 2g(W) - 2$$

thus $h^0(W, L_{|W}) > 0$ by R–R formula. □

We have to make the last preliminary to the proof of Theorem 3.8.1. Till now we always worked with LC pairs. Along the proof we will use pairs (X, D) which are not LC. To be able to treat this situation let us introduce the following definition and make some useful remarks.

Definition 3.8.7. The log canonical threshold related to a scheme $V \subset X$ of a pair (X, D) is $lct(X, V, D) := \inf\{t \in \mathbb{Q} : V \cap LLC(X, tD) \neq \emptyset\}$. We will say that (X, D) is LC along a scheme V if $lct(X, V, D) \geq 1$.

Remark 3.8.8. Let $Z \in CLC(X, lct(X, V, D)D)$ a center and assume that Z intersects V, then $(X, lct(X, V, D)D)$ is LC on the generic point of Z.
If (X, D) is not LC then Theorem 2.4.3 is in general false. On the other hand the first assertion stays true, also under the weaker hypothesis that (X, D) is LC on the generic point of $W_1 \cap W_2$. In fact the discrepancy is a concept related to a valuation ν, therefore we can always substitute the variety X by an affine neighborhood of the generic point of the center of ν.

Proof of theorem 3.8.1. Let $V = Bsl|L| \cap F$, remember that we are in a relative situation, therefore we need always to consider objects contained in a fixed fiber to fully enjoy the geometrical consequences of the ample anticanonical class.

Our aim is to derive a contradiction producing a section of L which is not identically vanishing along V. Consider the following set

$$\mathcal{D} := \{D \equiv_f \delta L : lct(X, V, D) = 1 \text{ and } dim W_D \leq r + 1 - \delta\},$$

where $W_D \in CLC(X, D)$ is a minimal center with $W_D \subset F$ and $W_D \cap V \neq \emptyset$.

Observe that $\mathcal{D}$ is non empty; consider in fact $D_0 = f^* \sum_I l_i(g_i)$, for g_i generic functions on S vanishing at $f(F)$ and $l_i \ll 1$.

Claim 1. There exists a $D \in \mathcal{D}$ such that $W_D \subset V$.

Proof of the claim. Consider the above D_0, $H \in |L|$ a generic section and

$$c = inf\{t \in \mathbb{Q} : LLC(D_0 + tH) \cap V \neq \emptyset\}.$$

Since H is a Cartier divisor vanishing on V, then $c \leq 1$. Let $D_c = D_0 + cH$. The minimality of W_{D_0} with respect to inclusion tell us that either $W_{D_0} \subset V$ or $c > 0$. Assume therefore that $c > 0$. If $c < 1$ then $W_{D_c} \subset V$ because H does not give any contribution on valuations outside of V. If $c = 1$ then both W_{D_0} and H are in $CLC(X, D_1)$, and their intersection is not empty because $W_D \cap V \neq \emptyset$. Therefore by Theorem 2.4.3 any irreducible component $Z \subset W_D \cap H$ is a center of dimension $dim Z = dim W_D - 1$. This means that $D_1 \in \mathcal{D}$ and $dim W_{D_1} < dim W_{D_0}$. Iterating this procedure we eventually produce D_{r+1} with $W_{D_{r+1}}$ a point in V. □

Let D be as in the claim, thus (X, D) is LC along W_D. If $r - \delta > -1$ then we can apply Lemma 3.8.6 to produce a section of L not vanishing along W_D and obtain a contradiction.

If $r - \delta = -1$ then W_D is a point in V and following the proof of the claim we deduce that

$$D = D_0 + \sum_1^{r+1} H_i,$$

with $H_i \in |L|$ generic.

Let $X_j = X \cap (\cap_1^j H_i)$, then X_j is LT in a neighborhood of $F_j := F \cap X_j$ for any $j \leq r + 1$. This assertion is left to be proved to the reader as an exercise. (hint: the main point to check is normality. To do it use the fact that terminal singularities are smooth in codimension 2.)

By vanishing theorem 2.3.2 we have the following surjection

$$H^0(X, L) \to H^0(X_j, L_{|X_j}),$$

for any $j \leq r$.

If the type of f is not $(1, 1, 1)$ then $f_r : X_r \to S$ is birational. In particular, by standard vanishing, $Z_r \simeq \mathbb{P}^1$. So that $L_{|Z_r}$ is spanned. The idea is to extend a section of $L_{|Z_r}$ not vanishing on Z_{r+1} to a section of $L_{|X_r}$. For details on this extension and about the case of type $(1, 1, 1)$ we refer to [Me3]. □

Part 4. Biregular geometry

Fano-Mori contractions are an essential tool in Minimal Model Program. We discuss their use in this frame in the first section of part 5. More generally they are important in problems of classification of projective varieties.

This part is devoted to the problem of describe F-M contraction. Unless for few results at the very beginning we will restrict ourself to the smooth case, that is we consider F-M contractions of smooth manifolds.

The singular case is very difficult and at the moment very few is known only in dimension 3 (essentially the complete classification of small extremal contractions on threefolds with at most terminal singularities in the fundamental paper of Mori [Mo4]).

We will give a complete classification of F-M contractions of smooth manifolds of dimension ≤ 4. We collect this classification in a sequel of theorems in the first section. We dedicate most of the part to their proof.

We are interested in a local description of the contraction, in a neighborhood of a given fiber; in particular we will consider a *local contraction around* F, $\varphi : X \to Z$, as defined in 3.6.9.

We will present many steps of the proof of the classification; each step is important by itself and together they represent a sort of program for classifying the F-M contractions. In short they are the following:

1) classify all possible fibers of the F-M contractions; we will succeed if their dimension is less or equal then two.

2) when the fiber has good singularities (locally complete intersections) classify the possible normal bundle of these fibers

3) describe a formal neighborhood of the possible fibers in X, i.e. the local contraction around F.

4) find a commutative diagram which includes φ and which can help in understanding φ (for example the flip in the small contraction case).

The results contained in this part are classical for the case $dim X = n = 2$, and they are due to the italian school of geometry of the beginning of the century.

In the case $n = 3$ they were proved by S. Mori in a famous paper, see [Mo3], who gave rise to the so called Mori theory.

The case $n = 4$ was later considered by M. Andreatta and J.A. Wiśniewski, see [AW3]. [AW2] is a survey of these results on which this part is strongly based.

In the last section we will outline the biregular classification of Fano manifolds of high index. These are the building block of F–M contractions and their knowledge is the starting point of any further investigation. Also in this case we will provide the known general techniques to approach the problem via adjunction methods, without any attempt to be exhaustive in the classification. In particular we will not present neither Fano–Iskovskikh approach based on double projections, [Is], nor Mukai vector bundle technique, [Mu], nor Ciliberto–Lopez–Miranda deformation ideas, [CLM].

4.9. Fano-Mori contractions on a smooth n-fold with $n \leq 4$

Here we describe all F-M contractions on smooth n-folds with $n \leq 4$. The case of dimension 4 is the more elaborate and it still presents some doubtful cases. More precisely there are some possible fibers which at the moment cannot be realized as fibers of a F-M contraction of a smooth 4-fold. Conjecturally they should not be realized in such a way.

Theorem 4.9.1. *Let X be a smooth projective surface and $R \subset \overline{NE(X)}$ an extremal ray such that $R \cdot K_X < 0$. Then the associated contraction morphism $cont_R : X \longrightarrow Z$ is one of the following:*

- *(1) Z is a smooth surface and X is obtained from Z by blowing-up a closed point; $\rho(Z) = \rho(X) - 1$.*
- *(2) Z is a smooth curve and X is a minimal ruled surface over Z; $\rho(X) = 2$.*
- *(3) Z is a point, $\rho(X) = 1$ and $-K_X$ is ample; in fact $X \cong \mathbb{P}^2$.*

Theorem 4.9.2. *Let X be a smooth projective 3-fold and $R \subset \overline{NE(X)}$ an extremal ray such that $R \cdot K_X < 0$. Then the associated contraction morphism $cont_R : X \longrightarrow Z$ is one of the following:*

- *(B) (Birational contractions) $dim Z = 3$, $cont_R$ is a divisorial contraction and there are five types of local behavior near the exceptional divisor E:*
  - *B1: $cont_R$ is the (inverse of the) blow-up of a smooth curve in the smooth threefold Z.*
  - *B2: $cont_R$ contracts a smooth $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-1)$; $cont_R$ is the (inverse of the) blow-up of a smooth point in the smooth threefold Z.*
  - *B3: $cont_R$ contracts a smooth two dimensional quadric, $\mathbb{F}_0$, with normal bundle $\mathcal{O}(-1)$; $cont_R$ is the (inverse of the) blow-up of an ordinary double point in Z (locally analytically, an ordinary double point is given by the equation $x^2 + y^2 + z^2 + w^2 = 0$).*
  - *B4: $cont_R$ contracts an irreducible singular two dimensional quadric, $\mathbf{S}_2$, with normal bundle $\mathcal{O}(-1)$; $cont_R$ is the (inverse of the) blow-up of a point in Z which is locally analytically given by the equation $x^2 + y^2 + z^2 + w^3 = 0$.*
  - *B5: $cont_R$ contracts a smooth $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-2)$; $cont_R$ is the (inverse of the) blow-up of a point in Z which is locally analytically given as the quotient of $\mathbb{C}^3$ by the involution $(x, y, z) \longrightarrow (-x, -y, -z)$.*
- *(C) (Conic Bundle) $dim Z = 2$ and $cont_R$ is a fibration whose fibers are plane conics (general fibers are of course smooth).*
- *(D) (del Pezzo fibration) $dim Z = 1$ and $cont_R$ is a fibration whose general fiber is a del Pezzo surface.*
- *(F) (Fano threefolds) $dim Z = 0$, $-K_X$ is ample, thus X is a Fano threefold, and $\rho(X) = 1$.*

The above two theorems work actually over an algebraically closed fields of any characteristic; the surface case follows from the fact that the Castelnuovo contraction theorem is true in any characteristic and the threefold one was proved by Kollár in [Ko1].

The next theorem aims to give the same result for the case $n = 4$; here the situation is much more intricated and it will take some space to be described.

The result comes from many contributions, the main ones are from Y. Kawamata, see [Ka2], and from M. Andreatta and J.A. Wiśniewski, see

[AW3] (in the fiber case, Y. Kachi obtained independently of [AW3] a similar classification of special two dimensional fiber of a conic fibration).

Theorem 4.9.3. *Let X be a smooth projective 4-fold and $R \subset \overline{NE(X)}$ an extremal ray such that $R \cdot K_X < 0$. Let $\varphi := cont_R : X \longrightarrow Z$ be the associated (local) contraction morphism, let $F = \varphi^{-1}(z)$ be a (geometric) fiber of φ (around which we will eventually shrink the morphism, see 3.6.9) and let E be the exceptional locus, where in the case φ is of fiber type we mean $E = X$.*

We divide the classification of these contractions by the couple of numbers $(dim E, dim \varphi(E))$ which we will call the signature of the contraction; note that the pair $(4, b)$ will be given to a fiber type contraction with $dim Z = b$ and the pairs (a, b) with $b \geq a$ cannot happen.

Part 0. *The are no F-M contraction of a fourfold of signature (a, b) with $a \leq 1$ and with $a = 2$ and $b = 1$.*

Part 1: Small contractions, see [Ka2]**.** *Let φ be a F-M contraction of a fourfold of signature $(2, 0)$. Then $E = F \simeq \mathbb{P}^2$ and its normal bundle is $N_{F/X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The contraction is completely determined in an analytic neighborhood by this two data, see 4.13.2 and also 4.13.1, and locally it is isomorphic to the contraction given by (see 3.7.5)*

$$\varphi : \mathbf{V}(\mathcal{E}) \to Z = Spec(\bigoplus_{k \geq 0} H^0(F, S^k(\mathcal{E}))$$

where $\mathcal{E} = \mathcal{O}(1) \oplus \mathcal{O}(1)$ and the map is associated to the evaluation of $S^k(\mathcal{E})$.

In this situation the flip of φ exists and it is obtained by blowing up E and then contracting the exceptional divisor in the other direction.

Part 2: Birational; divisor to point. *Let φ be a F-M contraction of a fourfold of signature $(3, 0)$. Then either E is $\mathbb{P}^3$, with normal bundle $\mathcal{O}(-a)$ and $1 \leq a \leq 3$, or a (possibly singular) three dimensional quadric, with normal bundle $\mathcal{O}(-a)$ and $1 \leq a \leq 2$, or otherwise $(E; -E_{|E})$ is a del Pezzo threefold, that is E has Gorenstein singularities, $-E_{|E}$ is ample and $K_E = 2E_E$ (these variety have been classified by T. Fujita, see* [Fu2] *and* [Fu4]*).*

Part 3: Birational; divisor to curve. *Let φ be a F-M contraction of a fourfold of signature $(3, 1)$. Then*

- *(a) The curve $C := \varphi(E)$ is smooth and $\varphi : X \to Z$ is the blow-up of Z along C.*
- *(b) $g := \varphi_{|E} : E \to C$ is either a $\mathbb{P}^2$-bundle or a quadric bundle.*
- *(c1) If E is a $\mathbb{P}^2$-bundle then the normal bundle of each fiber in X is either $\mathcal{O}(-1) \oplus \mathcal{O}$ or $\mathcal{O}(-2) \oplus \mathcal{O}$; in particular all fibers of φ are reduced and with no embedded components. In the first case Z is smooth and φ is the smooth blow-up; in the second $C = Sing Z$ and Z is locally isomorphic to $S_2 \times \mathbb{C}$ where S_2 is the germ of singularity obtained by*

contracting the zero section in the total space of the bundle $\mathcal{O}(2)$ *over* $\mathbb{P}^2$.

- *(c2) If* E *is a quadric bundle then the general fiber is irreducible and isomorphic to a two dimensional, possibly singular, quadric. Isolated special fibers can occur and they are isomorphic either to a singular quadric or to a reduced but reducible quadric (i.e. union of two* $\mathbb{P}^2$ *intersecting along a line); in particular there are no special fibers which are isomorphic to a double plane. The normal bundle of each fiber is* $\mathcal{O}(-1) \oplus \mathcal{O}$. *Locally* Z *can be described as a hypersurface of* $\mathbb{C}^5$*; in the following table we give a list of possibilities for* $Z = V(g) \subset \mathbb{C}^5$ *according to the described combinations of general and special fibers. We choose coordinates* $(z_1, z_2, z_3, z_4, z_5)$ *such that* $C = \{z_1 = z_2 = z_3 = z_4 = 0\} \subset \mathbb{C}^5$.

N^0	*special fib.*	*gen. fib.*	$g =$ *analytic equation of* Z
(1)	$\mathbb{F}_0$	$\mathbb{F}_0$	$z_1^2 + z_2^2 + z_3^2 + z_4^2$
(2)	$\mathbf{S}_2$	$\mathbb{F}_0$	$z_1^2 + z_2^2 + z_3^2 + z_5^m z_4^2, m \geq 1$
(3)	$\mathbf{S}_2$	$\mathbf{S}_2$	$z_1^2 + z_2^2 + z_3^2 + z_4^3$
(4)	$\mathbb{P}^2 \cup \mathbb{P}^2$	$\mathbf{S}_2$	$z_1^2 + z_2^2 + z_3^3 + z_4^3 + z_3^2 z_5^m, m \geq 1$
(5)	$\mathbb{P}^2 \cup \mathbb{P}^2$	$\mathbb{F}_0$	$z_1^2 + z_2^2 + z_3^3 + z_4^3 + z_3^2 z_5^m + z_3 z_4 f(z_5)$ $+ z_4^2 g(z_5)$ *with* $z^m g(z) \neq \frac{f(z)^2}{4}$

Part 4: Birational; divisor to surface. *Let* φ *be a F-M contraction of a fourfold of signature* $(3, 2)$. *Generically the map is described by the part 1 of Theorem 4.10.1; in particular* Z *as well as* $S := \varphi(E)$ *are in general smooth and* φ *is a simple blow-down of the divisor* E *to the surface* $S \subset Z$.

However there can be some special two dimensional fibers F. *If this is the case then the scheme theoretic fiber structure over* F *is trivial, that is the ideal* $\mathcal{I}_F$ *of* F *is equal to the inverse image of the maximal ideal of* z, *that is* $\mathcal{I}_F = \varphi^{-1}(m_z) \cdot \mathcal{O}_X$.

Moreover the fiber F *and its conormal bundle* $\mathcal{I}_F/\mathcal{I}_F^2$ *as well as the singularity of* Z *and* S *at* z *can be described as follows*

F	$N^*_{F/X}$	*SingZ*	*SingS*
$\mathbb{P}^2$	$T(-1) \oplus \mathcal{O}(1)/\mathcal{O}$	*cone over* $\mathbb{Q}^3$	*smooth*
$\mathbb{P}^2$	$\mathcal{O}^{\oplus 4}/\mathcal{O}(-1)^{\oplus 2}$	*smooth*	*cone over a twisted cubic*
quadric	*spinor bundle from* $\mathbb{Q}^4$	*smooth*	$non - normal$

The quadric fiber can be singular, even reducible, and in the subsequent table we present a refined description of its conormal bundle. The last entry in the table provides information about the ideal of a suitable surface S*; a complete description of these ideals can be find in* [AW4].

quadric	*conormal bundle*	$\mathcal{I}(S)$ *in* $\mathbb{C}[[x,y,z,t]]$
$\mathbb{P}^1 \times \mathbb{P}^1$	$\mathcal{O}(1,0) \oplus \mathcal{O}(0,1)$	(xz, xt, yz, yt)
quadric cone	$0 \to \mathcal{O} \to N^* \to \mathcal{J}_{line} \to 0$	*generated by* 5 *cubics*
$\mathbb{P}^2 \cup \mathbb{P}^2$	$T_{\mathbb{P}^2}(-1) \cup (\mathcal{O} \oplus \mathcal{O}(1))$	*generated by* 6 *quartics*

Part 5: Conic bundle fibration with possibly special two dimensional fiber. *Let φ be a F-M contraction of a fourfold of signature $(4,3)$. Then φ is a fibration whose general fibers are plane conics; generically the map is described by the part 2 of Theorem 4.10.1. In particular Z is in general smooth.*

However, in this dimension, there can be some special isolated two dimensional fibers F; the possibilities for F are the following:

- *$F \simeq \mathbb{P}^2$ and $N^*_{F/X} \simeq \mathcal{O}^3/\mathcal{O}(-2)$ or $T\mathbb{P}^2(-1)$. The scheme fiber structure $\tilde{F}$ is reduced and Z is smooth at $z = \varphi(F)$.*
- *F is an irreducible quadric and $N^*_{F/X}$ is the pullback of $T\mathbb{P}^2(-1)$ via some double covering of $\mathbb{P}^2$. The scheme fiber structure $\tilde{F}$ is reduced and Z is smooth at $z = \varphi(F)$.*
- *The following other possibilities for F can occur:*

$$(\mathbf{S}_3, \mathcal{O}(1)), (\mathbb{F}_1, C_0 + 2f)\mathbb{P}^2 \cup \mathbb{P}^2, \mathbb{P}^2 \cup \mathbb{F}_0, \mathbb{P}^2 \cup_{C_0} \mathbb{F}_1,$$
$$\mathbb{P}^2 \cup \mathbf{S}_2, \mathbb{P}^2 \cup \mathbb{P}^2 \cup \mathbb{P}^2, \mathbb{P}^2 \cup_f (\mathbb{F}_0)_{C_0} \cup \mathbb{P}^2,$$

where any two components intersect along a line (indicated by a subscript, when needed),
and the exceptional case of $\mathbb{P}^2 \bullet \mathbb{P}^2$ when the two components intersect at an isolated point.

Part 6: del Pezzo and Mukai fibration and Fano fourfolds. *Let φ be a F-M contraction of a fourfold of signature $(4,d)$, with $d \leq 2$. Then φ is an equidimensional fibration over Z. If $d = 2$ the general fiber is a del Pezzo surface, if $d = 1$ then the general fiber is a Mukai variety, while if $d = 0$, $-K_X$ is ample, thus X is a Fano fourfold, and $\rho(X) = 1$.*

Let us add some remarks at this long theorem.

Remark 4.9.4. The case $(3,0)$ is not complete, in fact it contains many non existing cases. More precisely if E is smooth del Pezzo threefold one can easily construct a F-M contraction of a smooth fourfold of signature $(3,0)$) by taking (see 3.7.5)

$$\varphi : \mathbf{V}(\mathcal{E}) \to Z = Spec(\bigoplus_{k \geq 0} H^0(F, S^k(\mathcal{E}))$$

where $\mathcal{E} = -E_{|E}$ and the map is associated to the evaluation of $S^k(\mathcal{E})$.

If E is non normal then it is conjecture that there are no F-M contraction of a smooth fourfold with E as exceptional divisor; in section 3 of [Fu4] this

case is discussed deeply and a lot of limitation on E are given (see 4.11.11 and the following discussion).

If E is normal then a list of possible E was given in [Be] but this list contains many redundant case too.

The case (4.3) is also non complete. In particular we have examples of appropriate 2 dimensional fibers except for the cases $\mathbb{P}^2 \cup \mathbf{S}_2$, $\mathbb{P}^2 \cup \mathbb{P}^2 \cup \mathbb{P}^2$ and $\mathbb{P}^2 \cup_f (\mathbb{F}_0)_{C_0} \cup \mathbb{P}^2$; we believe these cases cannot occur.

4.10. Fano-Mori contractions on a smooth n-fold with fibers of small dimension

In this section we present two theorems which characterize some F-M contractions of a smooth projective variety in higher dimension.

The first is due to T. Ando, see [An], and it deals with F-M contractions with one dimensional fibers.

Theorem 4.10.1. *Let $\varphi : X \to Z$ be a (local) Fano-Mori contraction of a smooth variety X of dimension n. Let $F = \varphi^{-1}(z)$ be a fixed fiber and assume that $\dim F = 1$.*
(1) If φ is birational then F is irreducible, $F \simeq \mathbb{P}^1$, $-K_X \cdot F = 1$ and its normal bundle is $N_{F/X} = \mathcal{O}(-1) \oplus \mathcal{O}^{(n-2)}$. The target Z is smooth and φ is a blow-up of a smooth codimension 2 subvariety of Z.
(2) If φ is of fiber type then Z is smooth and φ is a flat conic bundle. In particular one of the following is true:
(i) F is a smooth $\mathbb{P}^1$ and $-K_X \cdot F = 2$, $N_{F/X} \simeq \mathcal{O}^{(n-1)}$;
(ii) $F = C_1 \cup C_2$ is a union of two smooth rational curves meeting at one point and $-K_X \cdot C_i = 1$, $(N_{F/X})_{|C_i} \simeq \mathcal{O}^{(n-1)}$, $N_{C_i/X} \simeq \mathcal{O}^{(n-2)} \oplus \mathcal{O}(-1)$ for $i = 1,\ 2$;
(iii) F is a smooth $\mathbb{P}^1$, $-K_X \cdot F = 1$ and the fiber structure $\tilde{F}$ on F is of multiplicity 2 (a non reduced conic); the normal bundle of $\tilde{F}$ is trivial while $N_{F/X}$ is either $\mathcal{O}(1) \oplus \mathcal{O}(-1)^{(2)} \oplus \mathcal{O}^{(n-4)}$ or $\mathcal{O}(1) \oplus \mathcal{O}(-2) \oplus \mathcal{O}^{(n-3)}$ depending on whether the discriminant locus of the conic bundle is smooth at z or not.

The above theorem was generalized to the case of a variety X with terminal Gorenstein singularities by Mori and Kollár (see [KM1], (4.9) and (4.10.1)) for $n > 3$.

The case of an extremal contraction of a 3-fold X with terminal non Gorenstein singularities is much more difficult; this was discussed in the celebrated paper of Mori [Mo4] and in [KM1].

The next theorem is a generalization of the above theorem of Ando in the frame of *adjunction theory of projective varieties* a very classical theory (see [CE]), which was reproposed and improved in modern time by A. J. Sommese and his school.

One of the goals of this theory is to describe varieties X polarized by an ample line bundle L by means of the Fano-Mori contraction supported by

$K_X + rL$ where r is the nef value of the pair (X, L). If X is smooth and $r \geq (n-2)$ then this goal is achieved and we refer the reader to the book [BS] for an overview of the theory, see [Me1] for the terminal case.

The next theorem shows that it is also achieved when the nef value is large with respect to the dimension of fibers of φ.

Theorem 4.10.2 ([AW1]). *Let $\varphi : X \to Z$ be a (local) Fano-Mori contraction of a smooth variety X and let $F = \varphi^{-1}(z)$ be a fiber. Assume that φ is supported by $K_X + rL$, with L a φ-ample line bundle on X.*

- *(1) If $dimF \leq (r-1)$ then Z is smooth at z and φ is a projective bundle in a neighborhood of F.*
- *(2) If $dimF = r$ then, after possible shrinking of Z and restricting φ to a neighborhood of F, Z is smooth and*
  - *(i) if φ is birational then φ blows a smooth divisor $E \supset X$ to a smooth codimension $r-1$ subvariety $S \supset Z$,*
  - *(ii) if φ is of fiber type and $dimZ = dimX - r$ then φ is a quadric bundle,*
  - *(iii) if it is of fiber type and $dimZ = dimX - r + 1$ then $r \leq dimX/2$, $F = \mathbb{P}^r$ and the general fiber is $\mathbb{P}^{(r-1)}$.*

The basic steps of the proofs of these theorems are worked out in the next sections, together with the proofs of the results in the previous section.

4.11. The fibers of a Fano-Mori contraction

In this section we will try to give more informations on the possible fibers of the F-M contractions. In particular we will classify all possible fibers of dimension less or equal then two

4.11.1. **Using the vanishing theorem.** .

We want to show how the vanishing theorem implies vanishing results on the fiber. Subsequently we will show how these results, via the computation of the Hilbert polynomial of the (normalization) of the fiber will imply a bound on the dimension of the fiber.

The proof of the following proposition can be found in [Mo3], 3.20, 3.25.1, [Fu2], 11.3, [An] and [AW3], 1.2.1.

Proposition 4.11.1 (Vanishing of the highest cohomology). *Let $\varphi : X \to Z$ be a local F-M contraction around F supported by $K_X + rL$ (see 3.6.9). Let F' be a subscheme of X whose support is contained in the fiber F of φ, so that $\varphi(F') = z$. If either $t > -r$ or $t = -r$ and $dimF > dimX - dimZ$ then*

$$H^{dimF}(F', tL_{|F'}) = 0.$$

Proposition 4.11.2. *In the assumption of the above proposition let also $X' \in |L|$ be the zero locus of a non-trivial section of L. Then we have*

$$H^{dim(F\cap X')}(F' \cap X', tL_{|F'\cap X'}) = 0$$

if either $t > -r+1$ or $t = -r+1$ and $dim(F \cap X') \geq dimX - dimZ$.

Proof. We like to give here a proof of the second proposition, the proof of the first is similar. Note that $H^i(X, tL) = 0$ for $i > 0$ and $t > -r$ by the theorem (2.3.2); moreover we also have $H^i(X, tL) = 0$ for $t = -r$ and $i \geq dimX - dimZ$, for the so called Grauert-Riemenschneider-Kollár vanishing theorem (see [KMM], theorems 1.2.4 and 1.2.7). Thus from the exact sequence

$$0 \to -L \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0$$

tensorized by tL we also have $H^i(X', tL_{X'}) = 0$ for $i > 0$ and $t > -r+1$ or $t = -r+1$ and $i \geq dimX - dimZ$.

Now let $\mathcal{I}_{F'\cap X'}$ be the ideal of $F' \cap X'$ in X' and consider the sequence

$$0 \to \mathcal{I}_{F'\cap X'} \otimes tL \to \mathcal{O}_{X'} \otimes tL \to \mathcal{O}_{F'\cap X'} \otimes tL \to 0.$$

Take the associated long exact sequence. Since $H^i(X', \mathcal{I}_{F'\cap X'} \otimes tL)_z = 0$ for $i > q := dimF \cap X'$, the map $H^q(tL_{X'}) \to H^q(F' \cap X', tL_{F'})$ is surjective and the proposition follows from what we have observed at the beginning. □

The following result is a direct consequence of the above Proposition; it was proved by T. Fujita, see [Fu1], following arguments of S. Mori and T. Ando.

Theorem 4.11.3. *Let $\varphi : X \to Z$ be a local F-M contraction around F supported by $K_X + rL$. Then*

$$dimF \geq (r-1)$$

and if $dimF \geq dimX - dimW$ then

$$dimF \geq \lfloor r \rfloor$$

(If the contraction is of type $(d(\varphi), \gamma(\varphi), \Phi(\varphi))$ we say something like $-\gamma(\varphi) \leq d(\varphi) \leq \Phi(\varphi)$).

Proof. The inequalities $d(\varphi) \leq \Phi(\varphi)$ follows from the semicontinuity.

Let S be a component of a fiber F of dimension s and let $g : W \to S$ its desingularization. By the above proposition 4.11.1 and the Leray spectral sequence for g, exactly as in Lemma 2.4 of [Fu1], we get

$$H^s(W, g^*(tL)) = 0.$$

On the other hand, since $g^*(L)$ is nef and big on W, by the Kawamata-Viehweg vanishing theorem we have that $H^i(W, g^*(tL)) = 0$ for $t \geq -r$ and $0 < i < s$. Moreover, since L is ample, we have also that $H^0(W, g^*(tL)) = 0$ for $t < 0$.

Consider now the Hilbert polynomial $\chi(t) := \chi(W, g^*(tL))$; it is a polynomial in t of degree equal to $dimW = dimS$. By what proved above χ is

zero for all integers t such that $0 > t > -r$; if $dimF > dimX - dimZ$ and r is an integer then χ is zero also for $t = r$. The inequalities follows then immediately since $deg\chi \geq \{number\ \ of\ \ its\ \ zeros\}$.

□

4.11.1.1. *Exercises-Examples.*

Exercise 4.11.4. *Let (X, L) be a polarized variety; $K_X + nL$ is nef except when $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$. Also, if $n \geq 3$, $K_X + (n-1)L$ is nef unless (X, L) is one of the following: $(\mathbb{P}^n, \mathcal{O}(1))$, X is the quadric in $\mathbb{P}^n$ and L is a hyperplane section, X is the projectivization of a rank n vector bundle over a smooth curve A and $L = \mathcal{O}(1)$.*

Hint. This result was proved in the paper [Fu1] with the use of the above Theorems; a completely different proof, which make use of the deformation of rational curves and which works in all characteristic, has been recently given in [Ka-Ko].

4.11.2. **Existence of a Ladder for a fiber of a F-M contraction; horizontal slicing.** Here we will develop in more details the method outlined in the introduction of section 3.8. With the use of the theorem 3.8.1 we will study the fibers of a F-M contraction inductively. More precisely if F is a fiber of a Fano-Mori contraction of sufficiently high dimension (with the respect to the index or equivalently with "small" difficulty) then we can construct a ladder for the pair (F, L_F) and prove that $\Delta(F, L) = 0$.

In order to do this we first start with a Bertini type theorem.

Proposition–Definition 4.11.5 ([AW1]). *(Horizontal slicing) Let $\varphi : X \to S$ be a local contraction around $\{F\}$ supported by $K_X + rL$. Let $X_k = \cap_1^k H_i$, with $H_i \in |L|$ generic divisors.*

i) *Let $\varphi_{|X_k} = g \circ \varphi_k$ the Stein factorization of $\varphi_{|X_k} : X_k \to S$ then $\varphi_k : X_k \to S_k$ is a morphism with connected fiber, around $\{F \cap (\cap_1^k H_i)\}$, supported by $K_{X_k} + (r-k)L_{|X_k}$ and S_k is affine. In particular if X_k is normal then φ_k is a local contraction.*

Assume that X has LT singularities, $r \geq \epsilon\gamma(\varphi)$ and $k \leq r + 1 - \epsilon\gamma(\varphi)$.

ii) *The singularities of X_k are not worse then that of X outside of $Bsl|L|$, and any section of L on X_k extends to a section of L on X.*

Proof. The proof is straightforward and can be found in [AW1], Lemma 2.6, or in [Me3], Lemma 1.6. □

Theorem 4.11.6. *Let $\varphi : X \to Z$ be a local F-M contraction around F supported by $K_X + rL$ of type $(d(\varphi), \gamma(\varphi), \Phi(\varphi))$. Let S be any component of F_{red} and ϵ a sufficiently small positive rational number. If $\Phi(\varphi) \leq 1 - \epsilon\gamma(\varphi)$ or the type of φ is $(d, 1, 1)$, with $d \leq 0$, then $\Delta(S, L_S) = 0$.*

If $\Phi(\varphi) \leq -\epsilon\gamma(\varphi)$ or if the type of φ is $(-1, 1, 0)$, then F_{red} is irreducible and isomorphic to $\mathbb{P}^{dimF}$.

Proof. We present the proof which is contained in the paper ([Me3], Theorem 2.17); this is a slight generalization of the proof in ([AW3], Proposition 4.2.1) and ([AW2], Theorem 1.10)

To prove the first part of the Theorem let S be an irreducible component of F_{red} of dimension s and let $\delta := L^s.S$. We have to prove that $h^0(S, L_S) \geq \delta + r + 1$. To do this we will prove that there are at least $\delta + r + 1$ independent sections of $H^0(X, L)$ not vanishing identically on S. By the propositions 4.11.5 and 3.8.1 we reduce to the case of a contraction $\varphi : X \to T$ with one dimensional fiber F. Then, by assumption, we can use again propositions 4.11.5 and 3.8.1 and go one step further with a section $H \in |L|$; $\varphi_{|H} : H \to T$ is finite and by 4.11.5 all section of $|L_{|H}|$ extend to sections of $|L|$ proving the thesis.

Finally assume that $\Phi(\varphi) \leq -\epsilon\gamma(\varphi)$ and assume, by contradiction, that the fiber has (at least) two irreducible components intersecting in a subvariety of dimension $t \leq (r-1)$. By the base point freeness of L, we can choose $t+1$ sections of L intersecting transversally in a variety with log terminal singularities and meeting the two irreducible components not in their intersection. By construction the map φ restricted to this variety has non connected fibers and this is in contradiction with 4.11.5. Similarly one can prove that $L^r \cdot F = 1$ (we can slice to points and still have the connectedness, but then we must have only one point...) and thus that $F = \mathbb{P}^{dimF}$. □

An immediate corollary in the case of two dimensional fiber is the following.

Corollary 4.11.7. *Let F be a two dimensional fiber of a F-M contraction $\varphi : X \to Z$ of a Gorenstein variety X and let F' be any component. Assume that φ is birational or that the general non trivial fiber has dimension* 1. *Then F' is normal and the pair $(F', L_{|F'})$ has sectional (and thus Fujita Δ)-genus* 0 *and therefore (see section 1.2) it is among the following:*
(1) $(\mathbb{P}^2, \mathcal{O}(e))$, *with* $e = 1, 2$,
(2) $(\mathbb{F}_r, C_0 + kf)$ *with* $k \geq r+1$, $r \geq 0$,
(3) $(\mathbf{S}_r, \mathcal{O}_{\mathbf{S}_r}(1))$ *with* $r \geq 2$.

Moreover F is Cohen-Macaulay unless the zero locus of a general section in $|L_F|$ is disconnected.

We will see however in the next subsection that not all the possibilities can occur if the domain X is smooth (or has very good singularities). Another type of argument is needed to get rid of some cases.

To conclude the section we will mention another Bertini type theorem which has to do with the sections of the supporting divisor of the F-M contraction.

Proposition–Definition 4.11.8 ([AW1]). *(Vertical slicing) Let $\varphi : X \to S$ be a local contraction supported by $K_X + rL$, with $r \geq -1 + \epsilon\gamma(\varphi)$. Assume that X has LT singularities and let h be a general function on S. Let $X_h =$*

$\varphi^*(h)$ then the singularities of X_h are not worse than these of X and any section of L on X_h extends to X.

4.11.2.1. *Exercises-Examples.*

Exercise 4.11.9. *Let (X, L) be a polarized variety; assume that X has good singularities (smooth, KLT) and that there exists a section $D \in |L|$ with good singularities (smooth, KLT) and with a Fano-Mori contraction $\varphi : D \to Z$.*

Find good assumptions for which the contraction can be lifted to a F-M contraction on X and for which this contraction determine the structure of X. For instance assume that D is a scroll or a quadric fibration.

The same when instead of L we have an ample vector bundle $\mathcal{E}$ (see [AO1]*)*

Exercise 4.11.10. *Find a local Fano-Mori contraction around F supported by $K_X + rL$ of type $(d(\varphi), \gamma(\varphi), \Phi(\varphi))$ with $\Phi(\varphi) > 1 - \epsilon\gamma(\varphi)$ and for which F is non normal. Can you find such an example with X smooth? (the example (1.18) in* [AW2]*).*

However there is the following

Conjecture 4.11.11. Let $\varphi : X \to Z$ be a Fano-Mori contraction of a manifold of dimension ≤ 4. Then all the fibers are normal.

Note that the conjecture, after Mori's and Andreatta-Wiśniewski's work, is open only for the case in which $\varphi : X \to Z$ is a birational Fano-Mori contraction of a manifold of dimension 4 which contracts an irreducible divisor E to a point. Moreover, by the paper [Fu4], E, which is a del Pezzo threefold, is (possibly) non normal only if $(-K_E)^3 = 7$, $Sing(E) \cong \mathbb{P}^2$, the normalization of E is $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(5))$ and some other conditions.

4.11.3. **Rational curves on the fiber of F-M contractions.** In this subsection we will use another fundamental feature of a Fano-Mori contraction in order to complete the classification of possible two dimensional fibers: namely the existence of rational curves in its fibers.

4.11.3.1. *General Facts.* We have in fact the following existence theorem due to Mori [Mo2], [Mo3] in the smooth case and extended by Kawamata to the log terminal case in [Ka0]. We recall that a rational curve is a curve whose normalization is $\mathbb{P}^1$. (Although we work over $\mathbb{C}$, we would like to note that if X is smooth then the existence theorem is true also in positive characteristic; this concerns also the subsequent results obtained via deformation methods.)

Theorem 4.11.12 ((Existence of rational curves))**.** *Let $\varphi : X \to Z$ be a Fano-Mori contraction of a variety with log terminal singularities. Then the exceptional locus of φ is covered by rational curves contracted by φ.*

In this section we study deformations of rational curves following ideas started with the paper of Mori [Mo2]. We discuss only some of the results, concentrating on the case of smooth X. We refer the reader to the book of Kollár [Ko3] for general results concerning deformation of curves. The following result is from [Ko3], II.1.14.

Theorem 4.11.13. *Let C be a (possibly reducible) connected curve such that $H^1(C, \mathcal{O}_C) = 0$ and assume that C is smoothable (see* [Ko3], *II.1.10 for the definition; a tree of smooth rational curves, for instance, is smoothable, see* [Ko3], *II.1.12). Suppose that $f : C \to X$ is an immersion of C into a smooth variety X. Then any component of the Hilbert scheme containing $f(C)$ has dimension $-K_X \cdot C + (n-3)$ at least.*

The above result has several different versions. For example, Mori [Mo2] proved a version of it for maps of rational curves with fixed points. An important part of the Mori's proof of the existence of rational curves is a technique of deforming rational curves with a fixed 0-dimensional subscheme (to "bend" these curves) in order to produce rational curves of lower degree with respect to a fixed ample divisor (to "break" them). In short: if a rational curve can be deformed inside X with two points fixed then it has to break.

Mori's bend-and-break technique was used by Ionescu and Wiśniewski (see [Io], 0.4, and [Wi], 1.1) to prove a bound on the dimension of the fiber. The reader can compare this bound with the one obtained in Theorem 4.11.3.

Theorem 4.11.14. *Let $\varphi : X \to Z$ be a Fano-Mori contraction of an extremal ray R of a smooth variety X. Let E be the exceptional locus of φ (if φ is of fiber type then $E := X$) and let S be an irreducible component of a (non trivial) fiber F. Let $l = min\ \{-K_X \cdot C$: C is a rational curve in $S\}$.*

Then $dim S + dim E \geq dim X + l - 1$.

Corollary 4.11.15. *Let $\varphi : X \to Z$ be a Fano-Mori contraction of a smooth variety X supported by $K_X + rL$. Let E be the exceptional locus of φ and let S be an irreducible component of a (non trivial) fiber F.*

Then $dim S + dim E \geq dim X + r - 1$.

Moreover, in the assumption of the above corollary, if the equality holds then the normalization of S is $\mathbb{P}^s$; this is [AW3, Proposition (2.5)].

Thus one can propose the following conjecture

Conjecture 4.11.16. In the hypothesis of Theorem 4.11.14, if equality holds for an irreducible component S then $S = \mathbb{P}^s$.

Theorem 4.11.17. *If the contraction $\varphi : X \to Z$ is divisorial then the conjecture is true and φ is actually a smooth blow-up (i.e.of a smooth submanifold of Z which is also smooth).*

This theorem has recently been proved in [AO2].

A weaker (and still completely unsolved!) version of the above conjecture is

Conjecture 4.11.18. If X is a smooth projective variety of dimension n such that $K_X \cdot C \leq -n - 1$ for any complete curve $C \subset X$ then $X \simeq \mathbb{P}^n$.

4.11.3.2. *Rational curves on fibers of a F-M contraction of dimension* ≤ 2. Let us now try to work out a complete classification of fiber F of dimension ≤ 2 of a F-M contraction of a smooth variety X.

Lemma 4.11.19. *If a fiber F of a Fano-Mori contraction of a smooth n-fold X contains a component of dimension 1 then F is of pure dimension 1 and $-K_X \cdot F \leq 2$. In particular F is a line or a conic (with respect to the relative very ample line bundle $-K_X$), the last possibly reducible or non reduced. If the contraction is birational then F is a line.*

Proof. Let F' be a 1-dimensional component of F (F' is a rational curve because of 4.11.6). Then, by 4.11.13 $dim_{[F']}HilbX \geq -K_X \cdot F' + (n-3)$ and therefore small deformation of F' sweep out at least a divisor. More precisely: taking a small analytic neighborhood of $[F']$ in $Hilb$ and the incidence variety of curves we can produce an analytic subvariety $E \subset X$ which is proper over Z such that $F \cap E = F'$ and $dimE \geq n-1$. This implies that all components of F meeting F' are of dimension 1 and by connectedness of F we see that F is of pure dimension 1. The bound on the degree can be obtained similarly, (note that because of the base point freeness theorem $-K_X$ is φ-very ample so that one can apply 4.11.13 to a curve consisting of two components). □

Suppose now that $\varphi : X \to Z$ is a local Fano-Mori contraction of a smooth variety and F is an *isolated* fiber of φ of dimension ≥ 2; isolated means that all the neighboring fibers are of dimension ≤ 1. That is, because Z is affine, all the fibers of φ except F are of dimension ≤ 1.

Note that by the base point freeness theorem $L := -K_X$ is φ-very ample (see 3.8.1; the theorem states only the relative base point freeness of L, but as noticed in [AW3], Proposition 1.3.4, after possibly shrinking the affine variety Z, the same proof yields the relative very ampleness of L).

By Lemma 4.11.19 all 1-dimensional fibers of φ are of degree 1 (lines), or ≤ 2 (conics), with respect to $-K_X$, if φ is birational or of fiber type, respectively.

Let now $C \subset F$ be a rational curve or an immersed image of a smoothable curve of genus 0. If the degree of C with respect to $-K_X$ is bigger than that of 1-dimensional fibers of φ, then deformations of C **must remain inside** F which, in view of 4.11.13, provides us with the following useful observation:

Lemma 4.11.20. *In the above situation*

$$dim_{[C]}Hilb(F) \geq -K_X \cdot C + (n-3).$$

Let us explain why this simple observation is useful for understanding the structure of the fiber F. Lemma 4.11.20 can in fact be used to rule out many redundant case in the list 4.11.7 of possible components of F. We note

that the very ampleness of $L = -K_X$ as well as the precise description of the components of the fiber in 4.11.7 allow us to choose properly the curve which satisfies the assumptions in 4.11.20.

We will give just an example which explains our argument. All possible cases are discussed in details in [AW3], section 4.

Suppose that S is a component of F and $S \simeq \mathbf{S}_r$ where $r \geq 3$. Then as the curve C we take the union of general $r+1$ lines passing through the vertex of $\mathbf{S}_r$ (the lines are general so that none of them is contained in any other component of F). Then Lemma 4.11.20 implies that $\mathbf{S}_r$ can not be a component of F, for $r \geq 3$ if φ is birational and for $r \geq 4$ if φ is of fiber type.

Also this way, using 4.11.20, for a reducible fiber F we can limit the possible combination of irreducible components of F. To show how let us consider the following situation.

Lemma 4.11.21. *Let F be an isolated fiber of dimension ≥ 2 of a birational contraction $\varphi : X \to Z$ of a smooth n-fold X. Suppose that the exceptional locus of φ is covered by lines (with respect to $-K_X$). If there exists a nontrivial decomposition $F = F_1 \cup F_2$ then $F_1 \cap F_2$ does not contain 0-dimensional components.*

Proof. Let $x \in F_1 \cap F_2$ be an isolated point of the intersection. Since X is smooth $dim_x F_1 + dim_x F_2 \leq n$. For $i = 1,\ 2$ let $C_i \subset F_i$ be a line containing x. Then the variety parameterizing deformations of C_i inside of F_i with x fixed is of dimension $\leq dim_x F_i - 1$. Let us take $C = C_1 \cup C_2$. Then

$$dim_{[C]} Hilb(F) \leq dim_x F_1 + dim_x F_2 - 2 \leq n - 2$$

and because $-K_X \cdot C = 2$ we arrive to the contradiction with 4.11.20. □

Remark 4.11.22. Let us note that the above conclusion of 4.11.21 is no longer true if we do not assume that φ is birational, see [AW2, Example2.11.2].

With a combination of such arguments, all based on the Lemma 4.11.20, in the section 4 of [AW3], the following has been proved.

Proposition 4.11.23 ([AW2], Sect. 4)). *Let $\varphi : X \to Z$ be a Fano-Mori contraction of a smooth n-fold X with an isolated 2-dimensional fiber F; let $L = -K_X$.*

If φ is birational we have the following possibilities for the pair (F, L_F)

$n \geq 5$	$n = 4$	$n = 3$
$(\mathbb{P}^2, \mathcal{O}(1))$	$(\mathbb{P}^2, \mathcal{O}(1))$	$(\mathbb{P}^2, \mathcal{O}(1))$
$(n \leq 6)$	$(\mathbb{F}_0, C_0 + f)$	$(\mathbb{P}^2, \mathcal{O}(2))$
	$(\mathbf{S}_2, \mathcal{O}_{\mathbf{S}_2}(1))$	$(\mathbf{S}_2, \mathcal{O}_{\mathbf{S}_2}(1))$
	$(\mathbb{P}^2 \cup \mathbb{P}^2, \mathcal{O}(1))$	$(\mathbb{F}_0, C_0 + f)$
		$(\mathbb{F}_1, C_0 + 2f)$
		$\mathbb{P}^2 \cup_{C_0} \mathbb{F}_2$,
		with $L_{\mid \mathbb{P}^2} = \mathcal{O}(1), L_{\mid \mathbb{F}_2} = C_0 + 3f$

If φ is of fiber type and L is φ-spanned then we have the following possibilities for the pair (F, L_F)

$n \geq 5$	$n = 4, irreducible$	$n = 4, reducible$	$n = 3$
$(\mathbb{P}^2, \mathcal{O}(1))$	$(\mathbb{P}^2, \mathcal{O}(1))$	$\mathbb{P}^2 \cup \mathbb{P}^2$	$(\mathbb{F}_0, C_0 + 2f)$
$(n \leq 7)$	$(\mathbb{P}^2, \mathcal{O}(2))$	$\mathbb{P}^2 \bullet \mathbb{P}^2$	$\mathbb{F}_0 \cup \mathbb{F}_1$,
$(\mathbb{F}_0, C_0 + f)$	$(\mathbf{S}_2, \mathcal{O}(1))$	$\mathbb{P}^2 \cup \mathbb{F}_0$	$L_{\mathbb{F}_0} = C_0 + f$
$(n = 5)$	$(\mathbf{S}_3, \mathcal{O}(1))$	$\mathbb{P}^2 \cup_{C_0} \mathbb{F}_1$	$L_{\mathbb{F}_1} = C_0 + 2f$
$(\mathbb{P}^2 \cup \mathbb{P}^2, \mathcal{O}(1))$	$(\mathbb{F}_1, C_0 + 2f)$	$\mathbb{P}^2 \cup \mathbf{S}_2$	
$(n = 5)$	$(\mathbb{F}_0, C_0 + f)$	$\mathbb{P}^2 \cup \mathbb{P}^2 \cup \mathbb{P}^2$	
		$\mathbb{P}^2 \cup_{C_0} \mathbb{F}_0 \cup_f \mathbb{P}^2$	

In the above list the components of reducible fibers have a common line (in some cases we point out which line is it) with the unique exception of two $\mathbb{P}^2$s which meet at a point — we denote this union by $\bullet$. (We suppress the description of L whenever it is clear.)

Remark 4.11.24. Let us note that for almost all the above possibilities we can construct examples with appropriate isolated 2-dimensional fiber, see Section 3 of [AW3]. However, there are some exceptions for which we were unable to construct examples and we do not expect that all of them exist. This concerns only fiber type contractions and reducible fibers: $\mathbb{P}^2 \cup \mathbb{P}^2$ for $n = 5$ and $\mathbb{P}^2 \cup \mathbf{S}_2$, $\mathbb{P}^2 \cup \mathbb{P}^2 \cup \mathbb{P}^2$ and $\mathbb{P}^2 \cup \mathbb{F}_0 \cup \mathbb{P}^2$ for $n = 4$.

4.12. The description of the normal bundle of a fiber of a F-M contraction

In order to describe a contraction locally, after having determined the fiber, one has to find out what are the possible normal bundles of these fibers; of course when this is possible, that is when the fiber is a local complete intersection in X.

This can be considered as a second order type problem and it is very hard compared to the determination of the fiber. If the fiber is one dimensional it was considered by S. Mori in the case of 3-folds and by T. Ando in general.

The case of two dimensional fiber is one of the main achievement of the paper [AW3].

If the fiber is a divisor in X its normal bundle is already given by adjunction formula (since we know $K_{X|F}$); in general this gives only the first Chern class of the normal bundle.

Let us start with an easy lemma, which however gives a broader picture of what we are actually going to prove, namely the base point freeness of the normal bundle.

Lemma 4.12.1. [AW3, Proposition (3.5)] *Let $\varphi : X \to Z$ be a Fano-Mori or crepant contraction of a smooth variety with a fiber $F = \varphi^{-1}(z)$. Assume that F is locally complete intersection and that the blow up $\beta : \hat{X} \to X$ of X along F has log terminal singularities. By $\hat{F}$ we denote the exceptional divisor of the blow-up. Then the following properties are equivalent:*
(a) the bundle $N^*_{F/X}$ is generated by global sections on F,
(b) the invertible sheaf $\mathcal{O}_{\hat{X}}(-\hat{F})$ is generated by global sections at any point of $\hat{F}$.
(c) $\varphi^{-1}m_z \cdot \mathcal{O}_X = \mathcal{I}_F$ or, equivalently, the scheme-theoretic fiber structure of F is reduced, i.e. $\tilde{F} = F$.
(d) there exists a Fano-Mori contraction $\hat{\varphi} : \hat{X} \to \hat{Z} = Proj_Z(\bigoplus_k m_z^k)$ onto a blow-up of Z at the maximal ideal of z, and $\varphi^*(\mathcal{O}_{\hat{Z}}(1)) = \mathcal{O}_{\hat{X}}(1)$.

4.12.0.3. *The normal bundle of a 1-dimensional fiber.* The case in which F is a fiber of dimension 1 was mainly studied, after S. Mori, by T. Ando, [An]; we will report some of his results and we will introduce an alternative proof, as done in [AW3].

Let C be an irreducible component of F. As we have seen in Lemma 4.11.19, C is a rational curve and it can be either a line or a conic with the respect to $-K_X$. In the latter φ is of fiber type.

Let $\mathcal{I}$ be the ideal of $C \subset X$ (with the reduced structure) and consider the exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_X/\mathcal{I}^2 \to \mathcal{O}_X/\mathcal{I} \to 0.$$

In the long cohomology sequence associated the map of global sections $H^0(\mathcal{O}_X/\mathcal{I}^2) \to H^0(\mathcal{O}_X/\mathcal{I})$ is surjective. Moreover, by 4.11.1, we have the vanishing $H^1(\mathcal{O}_X/\mathcal{I}^2) = 0$. Therefore $H^1(\mathcal{I}/\mathcal{I}^2) = 0$ which gives a bound on $N^*_{C/X} = \mathcal{I}/\mathcal{I}^2$. Namely if $N_{C/X} = \oplus\mathcal{O}(a_i)$ then $a_i < 2$. On the other hand, by adjunction, $det(N_{C/X}) = \Sigma a_i = \mathcal{O}(-2 - K_X.C)$ and thus the list of possible values of $(a_1, ..., a_{n-1})$ are finite.

If φ is a good birational contraction then we have even a better bound because, similarly as above and using 4.11.1, we actually get $H^1(N^*_{C/X} \otimes \mathcal{O}(K_X.C)) = 0$. Therefore, since $K_X.C = 1$, there is only one possibility, namely $N_{C/X} = \mathcal{O}(-1) \oplus \mathcal{O}^{(n-2)}$.

If φ is of fibre type then the estimate coming from this technique is not sufficient and one has to use other arguments. More precisely, one has to deal with a scheme associated to a double structure on C — see [An].

It is also convenient to use arguments coming from the deformation theory. Namely, the possibilities which can occur from the above vanishing, if $n = 3$, are the following:

$$\mathcal{O} \oplus \mathcal{O}, \quad \mathcal{O} \oplus \mathcal{O}(-1), \quad \mathcal{O}(1) \oplus \mathcal{O}(-2), \quad \mathcal{O}(1) \oplus \mathcal{O}(-1).$$

We will show that the last possibility does not occur using an argument related to the deformation technique. It can be used to deal with 2 dimensional fibers too.

Lemma 4.12.2. *The normal bundle $N_{C/X}$ cannot be $\mathcal{O}(1) \oplus \mathcal{O}(-1)$.*

Proof. Assume the contrary and let $\psi : \hat{X} \to X$ be the blow-up of X along C. Let $E := \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}(-1))$ be the exceptional divisor. Let C_0 be the curve contained in E which is the section of the ruled surface $E \to C$ corresponding to the line bundle $\mathcal{O}(-1)$. We have immediately that $E \cdot C_0 = 1$ and that ψ_{C_0} is a $1-1$ map from C_0 to C. Therefore $K_{\hat{X}} \cdot C_0 = K_X \cdot C + E \cdot C_0 = -1$. In particular this implies that C_0 moves at least in a 1-dimensional family on $\hat{X}$ (see 4.11.13). Since it does not move on E it means that it goes out of E. Since C_0 is contracted by $\varphi \circ \psi$ it implies that $E.C_0 = 0$, but this is a contradiction since $E.C_0 = 1$. □

4.12.0.4. *The normal bundle of a two-dimensional fiber.* The case in which F is a fiber of dimension 2 was studied, by M. Andreatta and J.A. Wiśniewski, [AW3]. We will report here the main results contained in section 5.7 of [AW3], referring to it for proofs and more details.

Let us present in general the point of view of [AW3]. To understand higher dimensional fibers of Fano-Mori contractions we will slice them down. Thus we will need some kind of "ascending property".

Suppose that $\varphi : X \to Z$ is a Fano-Mori contraction of a smooth variety, $\mathcal{L}$ is an ample line bundle on X such that $-K_X - \mathcal{L}$ is φ-big and nef; for instance if φ is birational one can take $\mathcal{L} = L := -K_X$. Let $F = \varphi^{-1}(z)$ be a (geometric) fiber of φ. Suppose that F is locally complete intersection. Let $X' \in |\mathcal{L}|$ be a normal divisor which does not contain any component of F. Then the restriction of φ to X', call it φ', is a contraction, either Fano-Mori or crepant (see 4.11.5). The intersection $F' = X' \cap F$ is then a fiber of φ'. The regular sequence of local generators $(g_1, \ldots, g_r)$ of the ideal of the fiber F in X descends to a regular sequence in the local ring of X' which defines a subscheme $F \cdot X'$ supported on $F' = F \cap X'$, call it $\bar{F}'$. Let us note that if the divisor X' has multiplicity 1 along each of the components of F then, since a locally complete intersection has no embedded components, we get $\bar{F}' = F'$.

Lemma 4.12.3. *The scheme $\bar{F}'$ is locally complete intersection in X' and*

$$N^*_{\bar{F}'/X'} \otimes_{\mathcal{O}_{\bar{F}'}} \mathcal{O}_{F'} \simeq (N^*_{F/X})_{|F'}.$$

If moreover X' is smooth, $\mathcal{L}$ is spanned and $dim F' = 1$, then

$$H^1(F', (N^*_{F/X})_{|F'}) = 0$$

.

Proof. The first part of the lemma follows from the preceding discussion so it is enough to prove the vanishing. Let $\mathcal{J}$ be the ideal of $\bar{F}'$ in X'. From 4.11.1 we know that $H^1(\bar{F}', \mathcal{O}_{X'}/\mathcal{J}^2) = 0$ and since we have an exact sequence

$$0 \longrightarrow \mathcal{J}/\mathcal{J}^2 = N^*_{\bar{F}'/X'} \longrightarrow \mathcal{O}_{X'}/J^2 \longrightarrow \mathcal{O}_{X'}/\mathcal{J} = \mathcal{O}_{\bar{F}'} \longrightarrow 0$$

then we will be done if we show $H^0(\bar{F}', \mathcal{O}_{\bar{F}'}) = \mathbb{C}$. Since $H^1(\bar{F}', \mathcal{O}_{\bar{F}'}) = 0$ then this is equivalent to $\chi(\mathcal{O}_{\bar{F}'}) = 1$. The equality $H^0(\bar{F}', \mathcal{O}_{\bar{F}'}) = \mathbb{C}$ is clear if $\bar{F}'$ is reduced. But since $\mathcal{L}$ is spanned and F is locally complete intersection then there exists a flat deformation of $\bar{F}'$ to another intersection $F \cdot X''$ which is reduced. This is what we need, because flat deformation preserves Euler characteristic. □

Now let us consider the following ascending property. Take a point $x \in F'$. Suppose that the ideal of F', or equivalently $N^*_{F'/X'}$, is generated by global functions from X'. That is, there exist global functions $g'_1, \dots g'_r \in \Gamma(X', \mathcal{O}_{X'})$ which define F' at x. Then, since $H^1(X, -\mathcal{L}) = 0$ these functions extend to $g_1, \dots g_r \in \Gamma(X, \mathcal{O}_X)$ which define F. Thus passing from the ideal $\mathcal{I}$ to its quotient $\mathcal{I}/\mathcal{I}^2$ we get the first part of

Lemma 4.12.4. *If $N^*_{F'/X'}$ is spanned at a point $x \in F'$ by global functions from $\Gamma(X', \mathcal{O}_{X'})$ then $N^*_{F/X}$ is spanned at x by functions from $\Gamma(X, \mathcal{O}_X)$. If $N^*_{F'/X'}$ is spanned by global functions from $\Gamma(X', \mathcal{O}_{X'})$ everywhere on F' then $N^*_{F/X}$ is nef.*

Proof. We are only to proof the second claim of the Lemma. Since $F' \subset F$ is an ample section then the set where $N^*_{F/X}$ is not generate by global sections is finite in F. Therefore the restriction $(N^*_{F/X})_{|C}$ is spanned generically for any curve $C \subset F$ and consequently it is nef. □

If the fiber is of dimension 2 then we have a better extension property.

Lemma 4.12.5. *Let $\varphi : X \to Z$ be a Fano-Mori birational contraction of a smooth variety with a 2-dimensional fiber F which is a locally complete intersection. Let $L = -K_X$; it is a φ-ample line bundle which can be assumed φ-very ample (see 3.8.1 and* [AW3, Proposition (1.3.4)]*). Then the following conditions are equivalent:*
(a)$N^*_{F/X}$ is generated by global sections at any point of F
(b) for a generic (smooth) divisor $X' \in |L|$ the bundle $N^*_{F'/X'}$ is generated by global sections at a generic point of any component of F'.

Proof. The implication (a)⇒(b) is clear. To prove the converse we assume the contrary. Let S denote the set of points on F where $N^*_{F/X}$ is not spanned. Because of the extension property 4.12.4 and the fact that for a one dimensional fiber of a F-M contraction the spannedness of the normal bundle is equivalent to spannedness at a generic point (see [AW3, Corollary 5.6.2],), the set does not contain F' and thus it is finite. Now we choose another smooth section $X'_1 \in |L|$ which meets F along a (reduced) curve F'_1 containing a point of S. (We can do it because L is φ-very ample.) The bundle $N^*_{F'_1/X'_1}$ is generated on a generic point of F'_1 so it is generated everywhere but this, because of the extension property, implies that $N^*_{F/X}$ is generated at some point of S, a contradiction. □

Lemma 4.12.6. *Let $\varphi : X \to Z$ be a Fano-Mori birational contraction of a smooth 4-fold with a 2-dimensional fiber $F = \varphi^{-1}(z)$. As usually $L = -K_X$ is a φ-ample line bundle which may be assumed to be φ-very ample. Suppose moreover that either F is irreducible or $L^2 \cdot F \leq 2$ (which is the case when F is an isolated 2 dimensional fiber). Then the scheme fiber structure $\tilde{F}$ is reduced unless one of the following occurs:*
(a) the fiber F is irreducible and the restriction of N_F to any smooth curve $C \in |L_{|F}|$ is isomorphic to $\mathcal{O}(-3) \oplus \mathcal{O}(1)$,
(b) $F = \mathbb{P}^2 \cup \mathbb{P}^2$ and the restriction of N_F to any line in one of the components is isomorphic to $\mathcal{O}(-2) \oplus \mathcal{O}(1)$.

Proof. Let us consider a curve $C \in |L_{|F}|$. Since L is φ-very ample we can take a smooth $X' \in |L|$ such that $\varphi' = \varphi_{|X}$ is a crepant contraction and $C = F \cap X'$ (see 4.11.5). Then, considering the embeddings $C = F \cap X' \subset F \subset X$ and $C = F \cap X' \subset X' \subset X$, we get

$$N_{C/X} = N_{C/X'} \oplus L_C = (N_{F/X})_{|C} \oplus L_C$$

and therefore $N_{C/X'} = (N_{F/X})_{|C}$.

Now the normal bundles $N_{C/X'}$ of the crepant contraction φ' can be easily described, in a way similar to the previous section (see ([AW3], 5.6.1)). In particular it follows that if neither (a) nor (b) occurs then the fiber structure of the contraction φ' is reduced. Thus, using the previous lemma and the equivalence in 4.12.1, we conclude that $\tilde{F} = F$. □

Now, one has to discuss the possible exceptions described in the above lemma. This is done extensively in [AW3] and the following was proved:

Theorem 4.12.7. *(*[AW3]*, theorems 5.7.5 and 5.7.6) Let $\varphi : X \to Z$ be a Fano-Mori birational contraction of a smooth 4-fold with an isolated 2-dimensional fiber $F = \varphi^{-1}(z)$. Then the fiber structure $\tilde{F}$ coincides with the geometric structure F and the conormal bundle $N^*_{F/X}$ is spanned by global sections.*

Moreover if $F = \mathbb{P}^2$ *then* $N^*_{F/X}$ *is either* $\mathcal{O}(1) \oplus \mathcal{O}(1)$ *or* $T(-1) \oplus \mathcal{O}(1)/\mathcal{O}$, *or* $\mathcal{O}^{\oplus 4}/\mathcal{O}(-1)^{\oplus 2}$. *If* F *is a quadric (possibly singular or even reducible) then* $N^*_{F/X}$ *is the spinor bundle* $\mathcal{S}(1)$.

In some respects the above results about the fiber structure of a 2-dimensional fiber are nicer than one may expect. Namely, there is no multiple fiber structure, the conormal bundle is nef and the normal bundle of the geometric isolated fiber has no section. Thus the situation is better than for 1-dimensional isolated fibers in dimension 2 and 3: the fundamental cycle of a Du Val ADE surface singularity is non-reduced and in dimension 3 one may contract an isolated $\mathbb{P}^1$ with the normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(-3)$. On the other hand, using a double covering construction (see [AW2, Examples(3.5)]) in dimension 5 one may contract a quadric fibration over a smooth 3-dimensional base with an isolated fiber equal to $\mathbb{P}^2$, scheme theoretically the fiber is a double $\mathbb{P}^2$. Using the sequence of normal bundles and the deformation of lines argument, one may verify that in this case $N_F \simeq \mathcal{O}(1) \oplus \mathcal{E}^*$ where $\mathcal{E}$ is a rank 2 spanned vector bundle with $c_1 = 2$, $c_2 = 4$, so that $dim H^1(\mathcal{E}^*) = -\chi(\mathcal{E}^*) = 3$.

Let us also note that for divisorial fibers we have the following:

If $F = \bigcup F_i$ is a divisorial fiber of a surjective map $X \to Y$, where X is smooth and $dim Y \geq 2$ then for some $k > 0$ the line bundle $\mathcal{O}_{F_i}(-kF_i)$ has non-trivial section and thus no multiple of $\mathcal{O}_{F_i}(F_i)$ has a section. In particular, if $rank(Pic(X/Y)) = 1$ then $\mathcal{O}_F(-F)$ is ample.

One can then try to conjecture that if F is an isolated fiber of a (Fano-Mori) contraction which is locally complete intersection and with "small" codimension then $H^0(F, N_F) = 0$.

The above result on contractions of 4-folds can be generalized for the adjunction mappings of an n-fold. Namely, suppose that $\varphi : X \to Y$ is a Fano-Mori contraction of a smooth n-fold X supported by a divisor $K_X + (n-3)H$, where H is a φ-ample divisor on X. Since we are interested in the local description of φ around a non-trivial fiber $F = \varphi^{-1}(z)$, we may assume that the variety Z is affine.

Corollary 4.12.8. *Let us assume that* φ *is birational and that* F *is an isolated fiber of dimension* $n-2$. *If* $n \geq 5$ *then the contraction* φ *is small and* F *is an isolated non-trivial fiber. More precisely* $F \simeq \mathbb{P}^{n-2}$ *and* $N_{F/X} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, *and there exists a flip of* φ *(see [A-B-W]).*

Note that the preceding arguments, which led to the classification of the birational 4-dimensional case, depend on the isomorphism $\varphi'_* \mathcal{O}_{X'} \simeq \mathcal{O}_Z \simeq \varphi_* \mathcal{O}_X$. This fails to be true if φ is of fiber type. Namely, let $\varphi : X \to Z$ be a conic fibration, i.e. a Fano-Mori contraction such that $dim Z = dim X - 1$. As usually, we will assume that F is an isolated 2 dimensional fiber of φ and $L = -K_X$ is φ-spanned. Then the restriction of φ to a general section

$X' \in |L|$ is generically $2:1$ covering of Z. A different argument is developed for this case in [AW2], where the following theorem was proved.

Theorem 4.12.9 ([AW3], proposition 5.9.5 and theorem 5.9.6)**.**

Let $\varphi : X \to Z$ be a conic fibration of a smooth 4-fold. Suppose that F is an irreducible isolated 2-dimensional fiber of φ which is either a projective plane or a quadric. Then the conormal bundle $N^*_{F/X}$ is nef.

Moreover if $F \simeq \mathbb{P}^2$ then $N^*_{F/X} \simeq \mathcal{O}^3/\mathcal{O}(-2)$ or $T\mathbb{P}^2(-1)$. If F is an irreducible quadric then $N^*_{F/X}$ is the pullback of $T\mathbb{P}^2(-1)$ via some double covering of $\mathbb{P}^2$. In both cases the scheme fiber structure $\tilde{F}$ is reduced and Z is smooth at z.

4.13. When the normal of a fiber determines locally the contraction ?

We illustrate some results which show how the normal bundle can give all the informations we want on the contraction. That is when the second order approximation actually determines completely the formal neighborhood. This is of course not always the case.

In some situations the knowledge of the normal bundle $N_{F/X}$ allows to determine the singularity of Z at z. Let us recall that for a local ring $\mathcal{O}_{Z,z}$ with the maximal ideal m_z one defines the graded $\mathbb{C}$-algebra $gr(\mathcal{O}_{Z,z}) := \bigoplus_k m_z^k/m_z^{k+1}$. The knowledge of the ring $gr(\mathcal{O}_{Z,z})$ allows sometimes to describe the completion ring $\hat{\mathcal{O}}_{Z,z}$. Also, we will say that a spanned vector bundle $\mathcal{E}$ on a projective variety Y is p.n.-spanned (p.n. stands for projectively normal) if for any $k > 0$ the natural morphism $S^k H^0(Y, \mathcal{E}) \to H^0(Y, S^k\mathcal{E})$ is surjective. As we noted while discussing the contraction to the vertex, projective normality allows us to compare gradings of rings "upstairs" and "downstairs".

Proposition 4.13.1 ([Mo3])**.** *Let $\varphi : X \to Z$ be a contraction as above. Suppose moreover that $N^*_{F/X}$ is p.n.-spanned. Then $\varphi_*(\mathcal{I}_F^k) = m_z^k$, $\varphi^{-1}(m_z^k) \cdot \mathcal{O}_X = \mathcal{I}_F^k$ and there is a natural isomorphism of graded $\mathbb{C}$-algebras:*

$$gr(\mathcal{O}_{Z,z}) \simeq \bigoplus_k H^0(F, S^k(N^*_{F/X})).$$

We omit the proof of the above result referring to Mori, [Mo3], p.164, who proved it in case when F is a divisor, the generalization is straightforward.

The next is a version of a theorem of Mori [Mo3], 3.33, which is a generalization of a Grauert-Hironaka-Rossi result:

Proposition 4.13.2. *Suppose that F is a smooth fiber of a Fano-Mori or crepant contraction $\varphi : X \to Z$ and assume that its conormal bundle $N^* = N^*_{F/X}$ is nef. If $H^1(F, T_F \otimes S^i(N^*)) = H^1(F, N \otimes S^i(N^*)) = 0$ for $i \geq 1$ then the formal neighborhood of F in X is determined uniquely and it the*

same as the formal neighborhood of the zero section in the total space of the bundle N.

Also the following assertion is a straightforward generalization of the celebrated Castelnuovo contraction criterion for surfaces; its proof is similar to the one of [Ha], V.5.7, see also [AW2].

Proposition 4.13.3 (Castelnuovo criterion). *Let $\varphi : X \to Z$ be a projective morphism from a smooth variety X onto a normal variety Z with connected fibers (a contraction). Suppose that $z \in Z$ is a point of Z and $F = \varphi^{-1}(z)$ is the geometric fiber over z which is locally complete intersection in X with the conormal bundle $N^*_{F/X}$. Assume that for any positive integer k we have $H^1(F, S^k(N^*_{F/X})) = 0$ (note that this assumption is fulfilled if φ is Fano-Mori contraction, $N^*_{F/X}$ is nef and the blow-up of X at F has log terminal singularities). If for any $k \geq 1$ it is $S^k H^0(F, N^*_{F/X}) \simeq H^0(F, S^k(N^*_{F/X}))$ then z is a smooth point of Z and $dim Z = dim H^0(F, N^*_{F/X})$.*

4.14. Concluding remarks on the classification of Fano-Mori contractions on a smooth n-fold with $n \leq 4$

The proofs of the theorems announced in the first two sections of this part can be given applying the numerous results we have given up to now (very often not in a unique way). This may be not so trivial, so in this section we will give some possible schemes of proof.

A good starting point is to use the Ionescu-Wiśniewski inequality in 4.11.14, this will give the possibilities for the dimension of the fibers and of the exceptional locus. In particular it says that there are no small contractions (i.e. contractions whose exceptional locus has codimension ≥ 2) on a smooth threefolds and also it proves the part 0 of the Theorem 4.9.3.

Description of the F-M contractions around a fiber F of dimension 1 (general case).

This is given in 4.10.1; this cover almost all Theorem 4.9.1 and part of Theorems 4.9.2 and 4.9.3. The proof of 4.10.1 follows from 4.11.19 (which describes the possibilities for F), 4.12.2 and the preceding discussion (which describes the possibilities for the normal bundles; in 4.12.2 only the case $n = 3$ is discussed in details) and 4.13.3.

Note that in the case of surface if $\varphi = cont_R$ is a conic bundle then actually the surface is a minimal ruled surface. In fact if, by contradiction, F is a reducible or non reduced fiber of φ then $F = \sum a_i C_i = [C]$ with $[C] \in R$. But since R is extremal this implies that $C_i \in R$ for every i. Thus $C_i^2 = 0$, since a general fiber of φ is a smooth irreducible and reduced curve in the ray, and $C_i K_X < 0$. By the adjunction formula this implies that $C_i \simeq \mathbb{P}^1$ and $C_i K_X = -2$. Thus

$$-2 = (C \cdot K_X) = \sum a_i (C_i K_X) = -2 \sum a_i,$$

which gives a contradiction.

Description of the F-M contractions around a fiber F of dimension 2 (3-folds and 4-folds).

In the case of surface we have that the contraction φ contracts X to a point; that is that X has $Pic = \mathbb{Z}$ and $-K_X$ is a ample, i.e. X is a Fano surface of Picard number 1. It is a non trivial fact to prove directly that $X = \mathbb{P}^2$ (see the Exercise 4.15.12).

In the threefolds case we have that either φ is a contraction of fiber type contracting X to a curve, with all fiber of dimension two, or φ is a birational divisorial contraction which contracts a unique prime divisor equal to F to a point: in fact, by Exercise 3.6.8, φ cannot be a contraction to a surface with some isolated two dimensional jumping fibers and if φ is birational then the exceptional divisor is prime.

In the first case, by adjunction, the general fiber is a smooth del Pezzo surface.

In the birational case we can apply 4.11.23 which gives all the possibilities (namely 3) for the exceptional divisor F; by adjunction we easily compute the normal bundle and we prove the uniqueness of the analytic neighborhood of the contraction around F by using 4.13.3, 4.13.1 and 4.13.2. Actually for the uniqueness in the case of a fiber isomorphic to the singular quadric an extra argument is needed (see for this [Mo3, p. 165]). Note also that all these cases exist and can be constructed via the basic example 3.7.5 except the case with a fiber isomorphic to the singular quadric; in this case if we work as in 3.7.5 we construct a singular threefold X. A good example was given in [Mo3, 3.44.3].

We consider then Fano Mori elementary contraction from a fourfold, in a neighborhood of a two dimensional fiber F. We start with the birational case: the fiber can be an isolated two dimensional fiber or can stay in a one dimensional family of two dimensional fibers (the other possibilities are ruled out with Exercise 3.6.8).

The first case is described in Part 4 of the Theorem 4.9.3; to prove it we can first apply the Theorem 4.11.23, which gives 4 possibilities for the fiber F, namely F can be $\mathbb{P}^2$ or a reduced quadric (smooth, singular or reducible).

Then we apply the Theorem 4.12.7 which describes the possible normal bundles. If F is smooth then we can prove the uniqueness of a formal neighborhood of the fiber using 4.13.2 and construct an example using 3.7.5.

If F is a singular quadric or a reducible one (union of two $\mathbb{P}^2$ meeting along a line) then the situation is more complicated. In [AW4] one can find good examples for these situations; moreover in the case F is reducible there are two possible analytically non equivalent formal neighborhood of F. That is, as one expects, the fiber and its normal does not always determine the analytic neighborhood. The (open) question is whether in this case this two neighborhoods are the only possible ones (up to analytic equivalence).

Part 3 of the Theorem 4.9.3 describes the case in which F stays in a one dimensional family of two dimensional fibers. The proof of it is essentially different from the previous one and it was given in [AW4]. Using a vertical slicing, see the proposition 4.11.8, one can prove that the general two dimensional fiber of this family is either $\mathbb{P}^2$ or a irreducible and reduced quadric. In fact, in the notation of 4.11.8, $f_{X_h} : X_h \to f(X_h)$ is a Fano Mori contraction from a smooth threefold which contracts a general fiber to a point; thus we can apply the result on threefolds , i.e. 4.9.2. It is easy to prove that if the general fiber is $\mathbb{P}^2$ then the same is for the special fiber. If the general fiber is a quadric then the special one is also a quadric, but very likely more singular. It turns out that there are no non reduced quadric, that is double $\mathbb{P}^2$, as special fiber; the other possibilities all occur. We refer to [AW3] for further details and examples.

We finally pass to the case of fiber type Fano-Mori contractions with two dimensional fibers. If the two dimensional fiber is not isolated then the contraction is to a surface, by Exercise 3.6.8 it is equidimensional, i.e. all fibers are of dimension two, and its general fiber is a del Pezzo surface (by adjunction formula).

If the two dimensional fiber F is isolated then it is one of those described in the Theorem 4.12.7; not all the given possibilities are locally complete intersections. If F is a smooth $\mathbb{P}^2$ or a smooth quadric then the normal bundle is computed in the Theorem 4.12.9. In these two cases example can be constructed using 3.7.5; may be one can also prove the uniqueness of a formal neighborhood of the fiber using 4.13.2 (this can be a hard computation!). Some examples were constructed for other possible fibers (especially for the case $F = \mathbf{S}^3$ and $\mathbb{P}^2 \bullet \mathbb{P}^2$; in the last case the example is the one given in [AW4]) but for some of them we cannot even construct an example (see the remark after the Theorem 4.11.23.

Description of the F-M contractions around a fiber F of dimension 3 (4-folds).

If the contraction is birational by Exercise 3.6.8 F is the unique prime divisor equal to the exceptional locus. It is immediate to see, using adjunction formula, that F is a del Pezzo threefolds. The problem here is to prove that F is normal and which normal non smooth del Pezzo threefolds can actually occur (not all of them!) (see Part 2 of the Theorem 4.9.3 and the following remark).

If the contraction is of fiber type then, again by 3.6.8, all fibers are three dimensional and the generic one is a Mukai manifold.

4.15. Classification of Fano manifolds of high index

This section is devoted to the study of F-M contractions of a smooth manifolds with target a point, i.e. to Fano manifolds.

We already noticed that Fano varieties with high index, with respect to dimension, are easier to be understood. Namely for $i(X) \geq \dim X$ the

informations given by the Hilbert polynomial are already sufficient to give a complete description of all possible cases, see Exercise 3.8.5.

It is not surprising that for lower indices the world is wilder. The best known way to go further is the following adjunction procedure.

Let X be a Fano manifold of index $i(X) = r$ and fundamental divisor L (see the Definitions 3.8.3 and also 3.6.10). Assume that $|L|$ is not empty. Let $H \in |L|$ a generic member. Then by adjunction formula

$$-K_H = -(K + L)_{|H} \sim (r-1)L_{|H}.$$

In other words whenever $r > 1$ the section H is a Fano variety of the same dual index. So that, if one is able to control the singularities of H, then it is possible to study X trough H. More generally we can pose the following.

Definition 4.15.1. Let $f : X \to S$ a local contraction of type (d, γ, Φ), supported by $K_X + rL$. Then we will say that f has good divisors if, after maybe shrinking S, the generic element $H \in |L|$ has at worst the same singularities as X and $f_{|H} : H \to S_H$ is of type $(*, *, \Phi)$.

Assume that the good divisor problem has a positive answer for a fixed index. Then the above observation allows to classify all Fano manifolds of fixed dual index in an inductive way, starting from the lower dimensional ones. This is what Fujita did, see [Fu2], for del Pezzo manifold, i.e. Fano manifold with $i(X) = \dim X - 1$. His results can be summarized in the following way.

Theorem 4.15.2 ([Fu2]). *Let X be a del Pezzo manifold of dimension n. If $n = 2$ then X is either $\mathbb{Q}^2$ or $\mathbb{P}^2$ blown up in $r \leq 8$ general points. Assume $n \geq 3$. Let $d = H^n$ the degree of X. Then we have the following cases:*

$d = 1$ *$X_6 \subset \mathbb{P}(1^{n-1}, 2, 3)$, i.e. a hypersurface of degree 6 in the weighted projective space with weights $(1, \ldots, 1, 2, 3)$;*

$d = 2$ *$X_4 \subset \mathbb{P}(1^n, 2)$;*

$d = 3$ *$X_3 \subset \mathbb{P}^{n+1}$;*

$d = 4$ *$X_{2,2} \subset \mathbb{P}^{n+2}$, i.e. a complete intersection of two Quadrics*

$d = 5$ *a linear section of $\mathbb{G}(1, 4) \subset \mathbb{P}^9$;*

$d = 6$ *X is either $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2 \times \mathbb{P}^2$ or $\mathbb{P}_{\mathbb{P}^2}(T\mathbb{P}^2)$;*

$d = 7$ *the blow up of $\mathbb{P}^3$ in one point*

Before giving a brief outline of the proof in any dimension. We want to give a tutorial to the surface proof, entirely based on Minimal Model techniques.

Exercise 4.15.3. *Let S be a del Pezzo surface of degree d; then the Mori cone is spanned by extremal rays.*

- *Assume that S contains a (-1) curve C, let $\varphi : S \to S_1$ the contraction of C. Prove that S_1 is a del Pezzo surface of degree $d + 1$.*

So that after finitely many contractions we have a del Pezzo surface S_k with only fiber type extremal rays.

- *Show that S_k is either $\mathbb{Q}^2$ or $\mathbb{P}^2$.*
  hint: $h^1(S_k, \mathcal{O}) = 0$.

We have established that any del Pezzo surface S is either $\mathbb{Q}^2$ or the blow up of $\mathbb{P}^2$ in a finite number of points.

- *Show that you cannot blow up more than 8 points with the following restrictions: no 3 are on a line and no 6 on a conic.*
  hint: evaluate the self intersection of K_S and the intersection of K_S with the line or the conic.

We have now to conclude.

- *A surface obtained blowing up r points of $\mathbb{P}^2$, with the above restrictions, is a del Pezzo surface.*
  hint: Study either the combinatorics of the cone of effective curves, or the linear systems of cubics with imposed conditions.

One can improve the knowledge of these surfaces observing that whenever $d \geq 3$ then $|-K_S|$ is very ample and embeds $S \subset \mathbb{P}^d$. For for $d = 2$ $|-K_S|$ is spanned and gives a double cover of $\mathbb{P}^2$ ramified along a quartic. While for $d = 1$ $|-2K_S|$ is spanned and gives a double cover of a singular quadric ramified along a sextic and the vertex.

In any dimension the idea of the proof is to first show that the good divisor problem has a positive answer, see Exercise 4.15.12. Then all the information on del Pezzo surface can be extended. The first two cases can be obtained with the machinery of graduated rings, see [Mo1].

It is instructive to prove the less general fact that if a hyperplane section of a variety is a complete intersection then the variety itself is a complete intersection.

If degree $d \geq 3$ then $|-K_X|$ is very ample. So that $d = 3, 4$ are immediate. For higher degree the study is more subtle and we leave it for the interested reader, cfr [Fu2].

The next case is the one of Mukai manifold, i.e. manifold with $i(X) = \dim X - 2$. These varieties are named after S. Mukai who first announced their classification, [Mu], assuming the existence of good divisors.

Our aim is to apply the Base Point Free technique we have developed to answer the good divisor problem for Mukai varieties.

The idea is simple. Let X be a Fano manifold and $|L|$ the fundamental divisor of X. Let $D \equiv \delta L$ a $\mathbb{Q}$-divisor with $\delta < 1$. By BPF technique there is a section of $|L|$ non vanishing identically on $LLC(X, D)$. If we sum up with Bertini Theorem we immediately get that the generic section of $|L|$ cannot have worse than LC singularities. We actually prove the following.

Theorem 4.15.4 ([Me2]). *Let X be a Mukai variety with at worst log terminal singularities. Then X has good divisors except in the following cases:*

- *X is a singular terminal Gorenstein 3-fold which is a complete intersection of a quadric and a sextic in $\mathbb{P}(1, 1, 1, 1, 2, 3)$*

- *X is a terminal not Gorenstein 3-fold and the canonical cover of X is a complete intersection of a quadric and a quartic in $\mathbb{P}(1,1,1,1,1,2)$.*

In both exceptional cases the generic element of the fundamental divisor has canonical singularities.

Proof. We will prove the theorem in four steps, from the more singular to the smooth case. As usual we will argue by contradiction.

4.15.5 (X has log terminal singularities).

Proposition 3.8.4 ensures that $dim|L| \geq 1$. Let $H \in |L|$ a generic section and assume that H has worse then LT singularities. Let $\gamma = lct(X,H)$ and $Z \in CLC(X,\gamma H)$ a minimal center.

Exercise 4.15.6. *$\gamma \leq 1$ and $codZ \geq 2$.*

hint: if $codZ = 1$ then Z is a fixed component of $Bsl|L|$, but $dim|L| > 0$ and H is connected.

This is enough to derive a contradiction. By Bertini Theorem $Z \subset Bsl|L|$, while by Lemma 3.8.6 there exists a section of L non vanishing on Z.

4.15.7 (X has canonical singularities).

By the above H has LT singularities. Let $\mu : Y \to X$ a log resolution of (X,L), with $\mu^*H = \overline{H} + \sum r_iE_i$, where $|\overline{H}|$ is base point free, and $K_Y = \mu^*K_X + \sum a_iE_i$. Let us assume that H has not canonical singularities, then, maybe after reordering the indexes, we have $a_0 < r_0$. Since H is generic then $\mu(E_i) \subset Bsl|L|$, for all i with $r_i > 0$. Let $D = H + H_1$, with $H_1 \in |L|$ a generic section. First observe that μ is a log resolution of (X,D). Then (X,D) is not LC, indeed $a_0 + 1 < r_0 + r_0^1$, where $r_0^1 \geq 1$ is the multiplicity of H_1 at the center of the valuation associated to E_0. Let $\gamma = lct(X,D) < 1$ and W a minimal center of $CLC(X,\gamma D)$.

Exercise 4.15.8. *Prove that $codW \geq 3$.*

hint: H is KLT and canonical singularities are Gorenstein in codimension 2.

We are again in the condition to apply Lemma 3.8.6 to derive a contradiction.

4.15.9 (X has terminal singularities).

If X is a terminal Mukai variety of dimension≥ 4 then by the above Theorem we immediately get that the generic element H of the fundamental divisor $|L|$ is terminal. In fact outside $Bsl|L|$ it is terminal by Bertini Theorem and along the base locus the discrepancy must be positive, since the generic section of $L_{|H}$ is canonical.

What remains to be done is to study terminal 3-folds with $-K_X \equiv L$; for this we have the following.

Theorem 4.15.10 ([Me2]). *Let X be a terminal Mukai 3-fold, assume that all the divisor in the linear system $|L|$ are singular, then X is one of the two exceptions in the Theorem 4.15.4. They actually exist.*

4.15.11 (X is smooth).

Assume that the generic element in $|L|$ is not smooth then by the surjection

$$H^0(X, L) \to H^0(H, L_{|H}) \to 0,$$

and previous results $Bsl|L| = \{x\}$. Let $H_i \in |L|$, for $i = 1, \dots, n-1$ generic elements and $D = H_1 + \cdots + H_{(n-1)}$, then the minimal center of $CLC(X, D)$ is x and (X, D) is not LC at x, since $2(n-1) > n$. We, therefore, derive a contradiction by Lemma 3.8.6.

□

Exercise 4.15.12. *Let X be a del Pezzo manifold. Prove that X has good divisors.*

With similar arguments one can prove a good divisor problem for some F-M contractions, for details and related results see [Me3].

Theorem 4.15.13 ([Me3]). *Let $f : X \to S$ a local contraction of type $(1, 1, 1)$. Assume that X is smooth. Then f has good divisors.*

Remark 4.15.14. Could this be the starting point of a relative analogue of Fujita classification? The above theorem reduces this study to that of fibrations of surfaces. The main problem to solve is a base point free statement in a neighborhood of an irreducible non reduced fiber. With this one could provide a structure theorem as in the absolute case, embedding this spaces in some relative (weighted) projective space. Then one should try to extend Andreatta-Wiśniewski theory one step further. Unfortunately as far as we can say this is quite hard and requires lot of unknown results on vector bundles on del Pezzo surfaces.

Before closing this section we want to briefly review the main lines of the classification of Mukai manifold.

Let X be a smooth Fano n-fold of index $r = n - 2$. Let $|H|$ the linear system of fundamental divisors. The integer

$$g = \frac{1}{2} H^n + 1,$$

is called the genus of X and by Riemann–Roch

$$\dim H^0(X, H) = n + g - 1.$$

By Theorem 4.15.4 the generic element $S \in |H|$ is smooth. As observed above this allows an inductive argument towards three dimensional Fano's.

The classification is divided in two parts.

Part I: $rkPic(X) = 1$. By Iskovkikh's, [Is], results on Fano 3-folds we have

Proposition 4.15.15. *Let X be a smooth Mukai n-fold. Then $|H|$ is base point free and one of the following is true:*

i) *$|H|$ is very ample and embeds X in $\mathbb{P}^{g+n-2}$. In particular $X \subset \mathbb{P}^{g+n-2}$ has a smooth curve section canonically embedded.*

ii) *the morphism associated to sections of $|H|$ is a finite morphism of degree 2 either onto $\mathbb{P}^n$ (in case $g = 2$) or onto $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ (in case $g = 3$).*

By the above Proposition we are restricted to study projective varieties with smooth canonical curve section. The actual point of the classification is to understand all of them. For a somewhat backward approach, see also [CLM].

Theorem 4.15.16 ([Mu]). *Let $X_{2g-2} \subset \mathbb{P}^{n+g-2}$ be as in point i) of Proposition 4.15.15. If $g \leq 5$ then X is a complete intersection. Assume that $10 \geq g \geq 6$, then we have the following picture*

g	$n(g)$	$X_{2g-2}^{n(g)} \subset \mathbb{P}^{g+n(g)-2}$
6	6	$C(\mathbb{G}(1,4)) \cap \mathbb{Q} \subset \mathbb{P}^{10}$
7	10	$SO(10,\mathbb{C})/P \subset \mathbb{P}^{15}$
8	8	$\mathbb{G}(1,5) \subset \mathbb{P}^{14}$
9	6	$Sp(6,\mathbb{C})/P \subset \mathbb{P}^{13}$
10	5	$G_2/P \subset \mathbb{P}^{13}$

where $C(X)$ is the cone over X and $n(g)$ is the maximal dimension for a Mukai variety of that type (as observed before any hyperplane section is then a Mukai variety). If $g > 10$ then $g = 12$ and $n(g) = 3$. Then some special X_{22}^3 can be seen as a smooth equivariant compactification of $SO(3,\mathbb{C})$/(icosahedral group), [MU]. In general it is possible to give a description using net of Quadrics.

As before the case of genus≤ 5 is easily obtained, while the study of the remaining cases is the heart of the proof.
Part II: $rkPic(X) > 1$. In this case there are at least two extremal rays on X. One has to first study all the possibilities.

Exercise 4.15.17. *1) Prove that the only possible F-M birational contractions are those of a divisor to a point or to a smooth curve.*
2) Which divisors are then possible ?
3)In case of extremal rays with associated contraction of fiber type then the base of the contraction is smooth.

The information contained in the exercise are crucial to complete the biregular classification of all these varieties, [MoMu] [Mu].

Part 5. Birational geometry

One of the main goal of the Algebraic Geometry is to achieve a birational classification of projective algebraic varieties. The Minimal Model Program, or Mori's Program, is an attempt to get (part of) this classification. In the first section of this part we want to introduce the general philosophy of the MMP.

The techniques developed allow to treat other birational aspects as well. In this realm then we would like to focus on two different settings:#-Minimal Model and Sarkisov program.

The former is a polarized minimal model program that enable to study special uniruled 3-folds and it will be described in section 5.18.

The latter is used to investigate birational morphisms between Mori spaces, see Definition 5.16.6, and its application to the projective plane is outlined in section 5.17.

5.16. Minimal Model Program philosophy

Let us present the approach of the MMP towards the birational classification of Algebraic Varieties.

As we saw in the first part the case of smooth curves is clearly settled. The case of surfaces is more complicated and it was developed by Italians algebraic geometers at the beginning of the twentieth century. This big achievement was used as a model for the higher dimensional case and at the end it was included in the more general philosophy of Minimal Model Theory as we will see in the next.

Consider a smooth projective variety X, the aim of Minimal Model Theory is to distinguish, inside the set of varieties which are birational to X, a special "minimal" member $\tilde{X}$ so to reduce the study of the birational geometry of X to that of $\tilde{X}$. The first basic fact is therefore to define what it means to be minimal. This is absolutely a non trivial problem and the following definition is the result of hard work of persons like Mori and Reid in the late 70's, against the predominant philosophy of that time.

Definition 5.16.1. A variety $\tilde{X}$ is minimal if

- $\tilde{X}$ has $\mathbb{Q}$-factorial terminal singularities
- $K_{\tilde{X}}$ is nef

Let us make some observation on this definition. The first condition wants to express the fact that the minimal variety is *(semi) negatively curved.* We note in fact that if $detT_X = -K_X$ admits a metric with semi-negative curvature then K_X is nef. The converse is actually an open problem (true in the case of surfaces and may be it can be a conjecture).

Of course not always one expect to find a minimal model in the birational class of a given variety: there are variety which are non (semi) negatively curved. But in this case the MMP hopes to make a list of them; this works very well for surfaces as we will show in a while.

The condition on the singularities is the real break-out of the definition. The point of view should be the following: we are in principle interested in smooth varieties but we will find out that there are smooth varieties which does not admit smooth minimal models. However we can find such a model if we admit very mild singularities, the ones stated in the definition. Note also that terminal singularities are smooth in the surface case.

Starting from X we want therefore to determine a corresponding minimal model $\tilde{X}$. In view of 1.1.2 and 3.6.2 (or 3.6.5) the way to do it is quite natural. Namely, if K_X is not nef, then by 1.1.2 there exists an extremal ray (on which K_X is negative) and by 3.6.2 (or 3.6.5) we can construct an elementary (Fano-Mori) contractions $f : X \to X'$ which contracts all curves in this ray into a normal projective variety X'.

A naive idea at this point would be the following.

If f is of fiber type, i.e. $dim X' < dim X$, then one hope to recover a description of X via f; in fact, by induction on the dimension, one should know a description of X' and of the fibers of f. We will say something more on this case in the last part of the section (see 5.16.6).

If f is birational then one thinks to substitute X with X' and proceed inductively. Note that $rk(Pic(X')) > rk(Pic(X))$ and therefore this procedure has to get to an end, that is we should end up in a variety with K_X nef (or to a fiber type contraction).

The problem is of course that the Theorem 3.6.2 says very little about the singularity of the target,i.e. in the above notation of X'. It says only that it has normal singularities.

However in the surface case the situation is optimal, namely the theorem 4.9.1 applied in the above spirit gives immediately the following:

Theorem 5.16.2 (Minimal Model Program for surfaces). *Let S be a smooth surface. After finitely many blow downs of (-1)-curves one reach a smooth surface S' satisfying one of the following:*

1) *$K_{S'}$ is nef i.e. S' is a minimal model*
2) *S' is a ruled surface*
3) *$S' \simeq \mathbb{P}^2$.*

The main point is that if $cont_R$ is birational the the image is again a smooth surface (see 4.9.1). Then apply recursively 4.9.1. Thus to perform the MMP we can remain in the category of smooth surfaces. Moreover if $cont_R$ is of fiber type then its description is very precise.

In higher dimensions the requirement on the singularity starts to play. In particular we note that cases B3), B4) and B5) in the Theorem 4.9.2 lead to singular 3-folds. In particular the case B5) leads to a 2-Gorenstein singularity. However all this singularities are terminal and $\mathbb{Q}$-factorial. The fact that the Cone Theorem 1.1.2 and the Contraction Theorem 3.6.2 hold in the more general case of variety with terminal singularities seems to give some hopes.

Moreover the good property of birational contractions in the surface case ascends in higher dimension to the fact that if an elementary F-M contraction of a smooth (or terminal $\mathbb{Q}$-factorial) variety is divisorial then the target has at worst terminal $\mathbb{Q}$-factorial singularities, see Exercise 3.6.8.

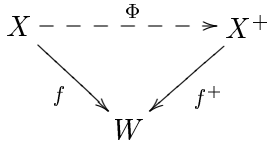
But a very serious problem is now coming up. Namely if we consider varieties with terminal singularities then they can have birational F-M contractions which are not divisorial! This was first noticed by P. Francia with a famous example, see for instance [CKM, p.33-34].

Let us see why an elementary contraction $f : X \to Y$ of a variety X with $\mathbb{Q}$-Gorenstein singularities and with exceptional locus E such that $codE \geq 2$ gives problems. Consider the sheaf $M = f^*(K_Y)_{|X\setminus E}$. Then $M \simeq K_{X|X\setminus E}$ and by the codimension assumption this yields $M^{**} \simeq K_X$. On the other hand $-K_X$ is f-ample while M is f-trivial. Thus K_Y is not $\mathbb{Q}$-Cartier!

In particular on such a Y even the definition of minimal model does not make sense. Our naive solution came abruptly to a stop and new ideas are needed. The main one is summarized in the following Flip conjectures, [KMM], which very roughly says that instead of contracting the exceptional locus of this "small" rays we have to make a codimension 2 surgery, called flip, that replace the curve with another which has a different normal sheaf.

This is the precise statement.

Conjecture 5.16.3 (Flip Conjecture). Let X be a terminal $\mathbb{Q}$-factorial variety and assume that there exists an extremal ray $\mathbb{R}^+[C] \subset \overline{NE}(X)$ with associated elementary contraction $f : X \to W$; assume also that $codim(E(f)) \geq 2$. Then there exists a terminal $\mathbb{Q}$-factorial variety X^+ together with the following commutative diagram



The following Theorem is the breakthrough of Mori Theory for threefolds which over-passed the flip problem. The proof is very intricated and it is based on a careful classification of all possible small contractions.

Theorem 5.16.4 ([Mo4]). *The Flip conjecture holds for threefolds.*

Then, assuming the flip conjecture, one asks if this sort of inductive procedure will come to an end, namely we need a kind of termination for these birational modification.

As noted above if f is a divisorial contraction then the Picard number drops by one so there cannot be an infinite number of those.

For flips there is not such a straightforward criterium and so the following termination conjecture raised.

Conjecture 5.16.5. Let X be a terminal $\mathbb{Q}$-factorial variety which is not minimal. Then after finitely many flips there is an extremal ray whose exceptional locus is of codimension≤ 1.

This conjecture is true for n-folds with $n \leq 4$, see [KMM, Theorem 5.1.15].

The other case when the above process gets to an end is when we encounter an elementary extremal contraction of fiber type. For this we give a definition.

Definition 5.16.6. A Mori space is a terminal $\mathbb{Q}$-factorial Fano-Mori contraction $\pi : X \to S$ such that $dimS < dimX$ and $rkPic(X/S) = 1$.

The goal in this case is, like in the surfaces, to get a classification of Mori Spaces.

In general the Mori space associated by the MMP is not uniquely determined. This problem arise when two extremal rays have not disjoint exceptional loci. A very simple example. Let $T = E \times \mathbf{F}_1$, where E is a smooth curve of genus $g > 0$. Then there are two extremal rays, one of divisorial type and the other of fiber type. In this case the order in which the rays are contracted determines the F-M space. In one case it is a $\mathbb{P}^1$-bundle in the other a $\mathbb{P}^2$-bundle.

Let us make some observation on these spaces. We note that the generic fiber is a variety with $\mathbb{Q}$-factorial terminal singularities with anticanonical bundle ample. Thus no multiple of the canonical bundle of X has a section, that is the Kodaira dimension of X is $-\infty$

Definition 5.16.7. A variety X is uniruled if there exists a generically finite surjective map $Y \times \mathbb{P}^1 \dashrightarrow X$.

Proposition 5.16.8. *Let $\pi : X \to S$ be a Mori Space; then X is a uniruled variety*

This proposition is a consequence of the following Theorem proved by using the theory of deformation of rational curves on smooth varieties (see [KMM, Corollary 5.1.4]).

Theorem 5.16.9 ([MiMo])**.** *Let X be a projective variety. Assume that for a general $x \in X$ there is a smooth proper curve C and a morphism $f : C \to X$ such that*

- *$x \in f(C)$*
- *X is smooth along $f(C)$*
- *$deg_C f^* K_X < 0$*

Then X is uniruled.

To conclude the above discussion let us write a Minimal Model Conjecture.

Conjecture 5.16.10. Let X be a projective variety with at most terminal singularities. Then there is a minimal model X' birational to X if and only if X is not uniruled.

In dimension 3 the above conjecture is now a Theorem. To prove it we need the following Theorem.

Theorem 5.16.11 ([Ko3])**.** *Let X be a smooth uniruled variety. Then there is a dense family of rational curves with negative intersection with the canonical class. In particular X has negative Kodaira dimension.*

Then the missing ingredient is the following exercise.

Exercise 5.16.12. *Let X be a threefold with terminal singularities covered by curves negative with respect to the canonical class. Then instead of reaching a minimal model, after finitely many birational modifications, we encounter an extremal ray whose exceptional locus covers the whole variety. In other words we reach a Mori space.*

This is true for n-folds as soon as we have the flip conjecture and the termination conjecture.

Summing things up we get.

Theorem 5.16.13. *Let X be a threefold with terminal singularities, then the Minimal Model conjecture holds.*

Let us also mention the following result.

Remark 5.16.14. The converse of Theorem 5.16.11 is a deep and challenging problem. It is conjectured that all smooth varieties with negative Kodaira dimension are uniruled. But a positive answer is only known up to dimension 3, as a byproduct of MMP, [Mi].

Therefore one can formulate the Minimal Model Conjecture in this form, which is also true in dimension 3.

Conjecture 5.16.15. Let X be a projective variety with at most terminal singularities. Then there is a minimal model X' birational to X if and only if $k(X) \neq -\infty$.

To conclude the section let us notice that a MMP for 4-folds is far to be reached at this stage. Theorem 4.9.3 is a first step in this direction. In this case one has small contractions also in the smooth case; but they are particularly easy and they can be flipped (see Part 1 of the Theorem 4.9.3).

5.17. The birational geometry of the plane

Sarkisov program is devoted to study the possible birational, not biregular, maps between Mori spaces. Here we will not try to outline the full program and its applications. For a complete introduction to aim and tools of Sarkisov Program we suggest [Co1], [Co2]. Instead we will give an idea of its techniques and possible applications in the simpler set up of surfaces. To do this we will prove the following Theorem using Sarkisov dictionary.

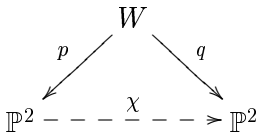
Theorem 5.17.1 (Noether-Castelnuovo)**.**
The group of birational transformations of the projective plane is generated by linear transformations and the standard Cremona transformation.

Let $\chi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ a birational map which is not an isomorphism. First we will factorize it in simpler maps, "elementary links", and then use them to prove the Theorem.

Let $\mathcal{H} = \chi_*^{-1}\mathcal{O}(1)$, the strict transform of lines in $\mathbb{P}^2$. Somehow the irregularity, "twisting", of the map χ is encoded in the base locus of H, $=: \operatorname{Bsl}\mathcal{H}$. Observe that $\mathcal{H}$ is without fixed components and $\mathcal{H} \subset |\mathcal{O}(n)|$ for some $n > 1$. Therefore the idea is to study the singularities of the log pair $(\mathbb{P}^2, \mathcal{H})$. The answer we are looking is the following

Theorem 5.17.2. *Let $\mathcal{H}$ as above then $(\mathbb{P}^2, 3/n\mathcal{H})$ has singularities which are worse that canonical singularities (we will say for short that the pair has not canonical singularities).*

The idea of the proof is as follows. Take a resolution



pull back the divisor $K_{\mathbb{P}^2} + 3/n\mathcal{H}$ and $K_{\mathbb{P}^2} + 3/n\mathcal{O}(1)$ via p and q respectively. Then play with the discrepancies. The crucial point is that on the right part you have some negativity coming from the non effective divisor $K_{\mathbb{P}^2}+3/n\mathcal{O}(1)$ that has to be compensated by some non effective exceptional divisor in the resolution p.

The above proof can be generalized to the following set up. Let $\pi : X \to S$ and $\varphi : Y \to W$ be two Mori spaces of dimension ≤ 3. Let $\chi : X \dashrightarrow Y$ a birational not biregular map. Choose $\mathcal{H}_Y$ a very ample linear system on Y. Let $\mathcal{H} = \chi_*^{-1}\mathcal{H}_Y$ then by the definition of Mori space there exists a $\mu \in \mathbb{Q}$ such that $K_X + 1/\mu\mathcal{H} \equiv_\pi 0$.

Theorem 5.17.3 (Noether–Fano inequalities)**.** [Co1] *If such a χ exists then either $(X, 1/\mu\mathcal{H})$ has not canonical singularities or $K_X + 1/\mu\mathcal{H}$ is not nef.*

The point of the above result is that the existence of the map χ imposes conditions on linear systems on X. Our first aim is to derive some consequence from it.

Let $\chi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ a birational not biregular map. Let $\mathcal{H} \subset |\mathcal{O}(n)|$ the linear system such that $(\mathbb{P}^2, 3/n\mathcal{H})$ is not canonical. $\mathcal{H}$ has not base components. Then there is a point $x \in \mathbb{P}^2$ such that

$$\operatorname{mult}_x \mathcal{H} > n/3. \tag{5.17.1}$$

Exercise 5.17.4. *Justify equation (5.17.1).*

Definition 5.17.5. A terminal extraction is a birational morphism with connected fibers $f : Y \supset E \to X \ni x$. Such that X and Y are terminal varieties. The exceptional locus is an irreducible divisor E, with $f(E) \ni x$, and $-K_Y$ is f-ample.

Exercise 5.17.6. *Prove that the only terminal extraction, from a smooth point of a surface is the blow up of the maximal ideal of a point. This is equivalent to inequality (5.17.1),* [Co1].

Let $\nu : \mathbb{F}_1 \to \mathbb{P}^2$ the blow up of x, with exceptional divisor C_0. First observe that the natural map $\pi_1 : \mathbb{F}_1 \to \mathbb{P}^1$ is a Mori space structure. Let $\chi' = \chi \circ \nu : \mathbb{F}_1 \dashrightarrow \mathbb{P}^2$ and $\mathcal{H}' = (\chi')^{-1}_* \mathcal{O}(1)$.

We are in the conditions to apply Theorem 5.17.3 to the linear system $\mathcal{H}'$.

Exercise 5.17.7. *Let $n' = n - mult_x \mathcal{H}$. Prove that $K_{\mathbb{F}_1} + 2/n' \mathcal{H}'$ is not canonical.*

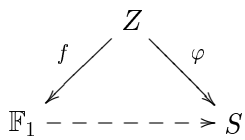
hint: either a direct computation or use the full strength of Noether–Fano inequalities.

So that the linear system $\mathcal{H}'$ admits a point $x' \in \mathbb{F}_1$ with "high multiplicity".

Let us go on with the same game: blow it up

$$f : Z \supset E \to \mathbb{F}_1 \ni x'.$$

This time Z is not a Mori space, but the fiber of $\mathbb{F}_1$ containing x' is now a (-1)-curve and we can contract it $\varphi : Z \to S$.



Exercise 5.17.8. *Prove that S is either a quadric, $\mathbb{F}_0$, or $\mathbb{F}_2$.*

hint: The case depends from the position of the point with respect to C_0.

This is known as an elementary transformation of ruled surfaces.

It is now a simple check that after finitely many transformations of this kind we reach a surface $\mathbb{F}_k$ together with a linear system $\mathcal{H}^r$. So that

$$K_{\mathbb{F}_k} + \frac{2}{n_r} \mathcal{H}_r \sim \pi_k^* A,$$

has canonical singularities and it is not nef.

In particular F_k is a del Pezzo surface.

Exercise 5.17.9. *In case $k = 1$ then what is left is to simply blow down the exceptional curve and reach $\mathbb{P}^2$ together with a linear system $\tilde{\mathcal{H}} \subset |\mathcal{O}(j)|$. Prove that this, after maybe a projectivity, is the end of the factorization.*

hint: Show that $K_{\mathbb{P}^2} + 3/j \tilde{\mathcal{H}}$ has canonical singularities and then apply NF-inequalities.

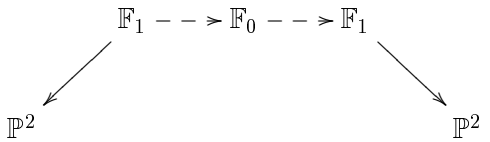
Sort it out what can be done when $k = 0$.

hint: Change the Mori space structure, keep in mind that $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Then show that you can do it only finitely many times.

We have factorized any birational, not biregular, self-map of $\mathbb{P}^2$ with a sequence of "elementary links". We will now use this information to prove the theorem we are aiming at.

The first step is to interpret a standard Cremona transformation in this new language

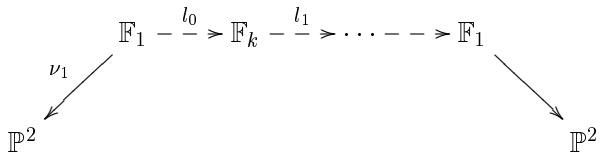
Exercise 5.17.10. *A standard Cremona transformation is given by the following links*



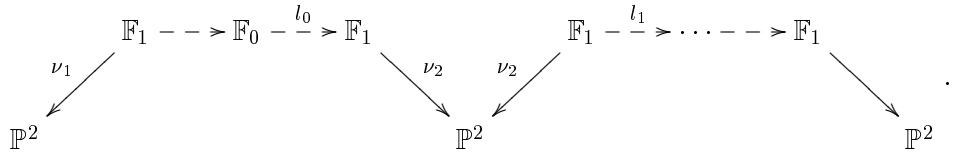
Vice-versa any link of this type can be factorized by Cremona Transformations. hint: Factorize the following map

$$(x_0 : x_1 : x_2) \to (x_1x_2 : x_0x_2 : x_1x_2 + x_0x_2 + x_0^2).$$

Let $\chi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ a birational map and



its factorization in elementary links. To conclude the proof we will force Cremona like diagram in it. Consider the left part of the factorization and substitute it, at the cost of introducing new singularities, with



Exercise 5.17.11. *Prove that after finitely many of the above modifications the map χ is factored by Cremona transformations.*

hint: One can consider sub-diagrams containing at most one $\mathbb{F}_0$. For such a diagram let

$$d = \max\{k : \text{there is an } F_k \text{ in that sub-diagram}\}.$$

Observe that after the substitution d is decreased.

5.18. #-Minimal Model

We already pointed out that Minimal Model Program allows to attach a Mori space to a uniruled 3-fold. Well how to use it to study the birational geometry of X ?

The main difficulty here is that the birational modifications occurring along the MMP are difficult to follow and usually it is almost impossible to guess what the output will be. We will rephrase, after [Re3], the standard minimal model program for uniruled varieties **but** using a polarizing divisor, it will be called #-minimal model. Under strong assumptions on the variety studied, we will be able to govern the program and understand its output.

Definition 5.18.1. Let T be a terminal $\mathbb{Q}$-factorial uniruled 3-fold and $\mathcal{H}$ a movable linear system, i.e. $\dim |mH| > 0$ for $m \gg 0$, with generic element $H \in \mathcal{H}$ on T. Assume that H is nef, then

$$\rho = \rho_{\mathcal{H}} = \rho(T, \mathcal{H}) =: \sup \{m \in \mathbb{Q} | H + mK_T \text{ is an effective } \mathbb{Q}\text{-divisor } \},$$

is the threshold of the pair $(T, \mathcal{H})$.

Since we are assuming that $dim \mathcal{H} \geq 0$ then $\rho \geq 0$. We will associate to $(T, \mathcal{H})$ a Mori fiber space $\pi : T^{\#} \to W$ and a movable Weil divisor $\mathcal{H}^{\#}$ with generic element $H^{\#} \in \mathcal{H}^{\#}$, such that

$$\rho(T, \mathcal{H}) K_{T^{\#}} + H^{\#} \equiv_{\pi} \mathcal{O}_{T^{\#}}.$$

This new pair $(T^{\#}, \mathcal{H}^{\#})$ will be called #-minimal model of $(T, \mathcal{H})$.

To find a #-minimal model of $(T, \mathcal{H})$ let us proceed in the following way.

Let $(T_0, \mathcal{H}_0) = (T, \mathcal{H})$, H_0 is nef by hypothesis and T_0 is uniruled; therefore to (T_0, H_0) is naturally associated the nef value $t_0 = sup\{m \in \mathbb{Q} | mK_{T_0} + H_0 \text{ is nef } \}$ and a rational map $\varphi_0 : T_0 \dashrightarrow T_1$, which is either an extremal contraction or (if the extremal contraction is small) a flip, of an extremal ray in the face spanned by $t_0 K_{T_0} + \mathcal{H}_0$ (see section (3.1) and in particular 3.6.7).

Consequently on T_1 one defines a movable linear system by $\mathcal{H}_1 := \varphi_{0*} \mathcal{H}_0$. That is to say $\varphi_{0*} H_0 \neq 0$.

Note that $t_{i-1} K_{T_i} + H_i$ is nef, thus one inductively defines $\varphi_i : T_i \dashrightarrow T_{i+1}$ and $(T_{i+1}, \mathcal{H}_{i+1})$ as follows.

Let $\delta = sup\{d \in \mathbb{Q} | dK_{T_i} + (t_{i-1} K_{T_i} + H_i) \text{ is nef } \}$ and define $t_i := \delta + t_{i-1}$.

Exercise 5.18.2. *Prove that there always exists an extremal ray $[C_i] \subset \overline{NE}(T_i)$ in the face spanned by $t_i K_{T_i} + H_i$.*

Thus let us define $\varphi_i : T_i \dashrightarrow T_{i+1}$ the birational modification associated to the extremal ray $[C_i] \subset \overline{NE}(T_i)$, or is flip if it is small, and $\mathcal{H}_{i+1} := \varphi_{i*} \mathcal{H}_i$.

The inductive process is therefore composed by divisorial contractions and flips and, since T_0 is uniruled, after finitely many of these we get a Mori fiber space.

Exercise 5.18.3. *Prove that the output $(T_k, \mathcal{H}_k)$ is a #-minimal model, that is*

$$\rho(T, \mathcal{H}) K_{T_k} + H_k \equiv_{\pi} \mathcal{O}_{T_k}.$$

Remark 5.18.4. Note that $\mathcal{H}^{\#}$ is relatively nef. Furthermore if the rational map defined by $|mH|$ is birational then $\mathcal{H}^{\#}$ is relatively ample.

In general we are not able to handle the whole #-program. It would be like handling a generic MMP. But the existence of the polarization allows to control the steps imposing restrictions on the threshold. Let $(T, \mathcal{H})$ as above and assume moreover that $\rho_{\mathcal{H}} < 1$ and that there is a smooth surface $S \in \mathcal{H}$. Under this assumption it is possible to describe in detail the #-process in a neighborhood of the surface S, see also [CF].

Proposition 5.18.5 ([Me4]). *Let $\varphi_i : T_i \dashrightarrow T_{i+1}$ a birational modification in the #-program relative to $(T, \mathcal{H})$. Assume that $S \in \mathcal{H}_i$ is a smooth surface. Then $\varphi_i(S) = \overline{S}$ is a smooth surface and $\varphi_{i|S} : S \to \overline{S}$ is either an isomorphism or the contraction of a disjoint union of (-1)-curves.*

Sketch of proof. Since S is smooth and T_i is terminal $\mathbb{Q}$-factorial then S is out of $Sing(T_i)$. In particular H_i is a Cartier divisor. We have the following cases.

- [φ_i contracts a divisor E onto a curve] Then $H_i \equiv_{\varphi_i} 0$ and $S \cap E$ is the disjoint union of (-1)-curves.
- [φ_i is a flip] S is disjoint from the flipping curve
- [φ_i is a divisorial contraction to a point] φ_S is birational and is either an isomorphism or the contraction of a (-1)-curve. $(E, E_{|E}) \simeq (\mathbb{P}^2, \mathcal{O}(-1))$ and $H_{|E} \sim \mathcal{O}(1)$.

□

Using the above Proposition we can control the # minimal model and its output.

Corollary 5.18.6. *Let T be a terminal $\mathbb{Q}$-factorial uniruled 3-fold and $\mathcal{H}$ a movable and nef linear system and $(T^\#, \mathcal{H}^\#)$ a #-minimal model of $(T, \mathcal{H})$. Assume that $\rho_{\mathcal{H}} < 1$ and $\mathcal{H}$ is base point free then $H^\# \in Pic(T^\#)$, $\mathcal{H}^\#$ has at most base points and $H^\#$ is smooth.*

Proof. By Bertini Theorem H is smooth therefore we can apply Proposition 5.18.5 in an inductive way up to reach a model $(T^\#, \mathcal{H}^\#)$. □

We will need the following relative version of Corollary 5.18.6.

Definition 5.18.7. Let T be a 3-fold and $\mathcal{H}$ a movable linear system, with $dim\mathcal{H} \geq 1$. Assume that $H = M + F$, where $\mathcal{M}$ is a movable linear system without fixed component and F is the fixed component. A pair $(T_1, \mathcal{H}_1)$ will be called a log minimal resolution of the pair $(T, \mathcal{H})$ if there is a morphism $\mu : T_1 \to T$, with the following properties:

- T_1 is terminal $\mathbb{Q}$-factorial
- $\mu_*^{-1} M = H_1$, where H_1 is a Cartier divisor, $dimBsl(\mathcal{H}_1) \leq 0$
- a general element $H_1 \in \mathcal{H}_1$ is a minimal resolution of a general element $M \in \mathcal{M}$.

Corollary 5.18.8. *For any pair $(T, \mathcal{H})$ with T an irreducible $\mathbb{Q}$-factorial 3-fold and $\mathcal{H}$ a movable linear system with $dim\mathcal{H} \geq 1$, there exists a log minimal resolution.*

Remark 5.18.9. If T is an irreducible uniruled 3-fold and $\mathcal{H}$ a movable linear system of Weil divisors, we will call #-Minimal Model of the pair $(T, \mathcal{H})$, a #-Minimal Model of a log minimal resolution. Note that this is well defined only up to birational equivalence.

Using Corollary 5.18.8 we can describe a pair $(T, \mathcal{H})$, with $dim\mathcal{H} \geq 1$. A #-Minimal Model of a log minimal resolution.

5.19. Applications of the #-program

We want now to apply the # theory to some concrete situation. Despite the assumption, needed to use the results of the previous section, are quite strong, they are geometric in nature and therefore of easy interpretation.

5.19.1. **3-folds with a uniruled movable system.**

Definition 5.19.1. Let T be a terminal $\mathbb{Q}$-factorial 3-fold and $\mathcal{H}$ a movable linear system. We will say that $(T, \mathcal{H})$ is a pair with a big uniruled system if $H \in \mathcal{H}$ is nef and big and H is a smooth surface of negative Kodaira dimension.

Exercise 5.19.2. *Let $(T, \mathcal{H})$ be a pair with a big uniruled system. Then T is uniruled and $\rho(T, \mathcal{H}) < 1$.*

Using #-MMP techniques we can describe in detail the possibilities that occur under these conditions.

Theorem 5.19.3 ([Me4])**.** *Let $(T, \mathcal{H})$ be a pair with a big uniruled system. Then $(T^{\#}, \mathcal{H}^{\#})$ is one of the following:*

i) *a $\mathbb{Q}$-Fano 3-fold of index $1/\rho > 1$, with $K_{T^{\#}} \sim -1/\rho H^{\#}$ and $\Phi_{|H^{\#}|}$ birational, the complete classification is given in* [CF] *and* [Sa]*:*
   $(\mathbb{P}(1,1,2,3), \mathcal{O}(6))$
   $(X_6 \subset \mathbb{P}(1,1,2,3,a), X_6 \cap \{x_4 = 0\})$, *with* $3 \leq a \leq 5$
   $(X_6 \subset \mathbb{P}(1,1,2,2,3), X_6 \cap \{x_3 = 0\})$
   $(X_6 \subset \mathbb{P}(1,1,1,2,3), X_6 \cap \{x_0 = 0\})$
   $(\mathbb{P}(1,1,1,2), \mathcal{O}(4))$
   $(X_4 \subset \mathbb{P}(1,1,1,1,2), X_4 \cap \{x_0 = 0\})$
   $(X_4 \subset \mathbb{P}(1,1,1,2,a), X_4 \cap \{x_4 = 0\})$, *with* $2 \leq a \leq 3$
   $(\mathbb{P}^3, \mathcal{O}(a))$, *with* $a \leq 3$, $(\mathbb{Q}^3, \mathcal{O}(b))$, *with* $b \leq 2$
   $(X_3 \subset \mathbb{P}(1,1,1,1,2), X_3 \cap \{x_4 = 0\})$, $(X_3 \subset \mathbb{P}^4, \mathcal{O}(1))$
   $(X_{2,2} \subset \mathbb{P}^5, \mathcal{O}(1))$
   a linear section of the Grassmann variety parametrising lines in $\mathbb{P}^4$, embedded in $\mathbb{P}^9$ by Plucker coordinates
   $(\mathbb{P}(1,1,1,2), \mathcal{O}(2))$, *the cone over the Veronese surface*
ii) *a bundle over a smooth curve with at most finitely many fibers $(G, H^{\#}_{|G}) \simeq (\mathbf{S}_4, \mathcal{O}(1))$, and generic fiber $(F, H^{\#}_{|F}) \simeq (\mathbb{P}^2, \mathcal{O}(2))$. Where $\mathbf{S}_4$ is the cone over the normal quartic curve and the vertex sits over an hyperquotient singularity of type $1/2(1,-1,1)$ with $f = xy - z^2 + t^k$, for $k \geq 1$,* [YPG]*,*
iii) *a quadric bundle with at most cA_1 singularities of type $f = x^2 + y^2 + z^2 + t^k$, for $k \geq 2$, and $H^{\#}_{|F} \sim \mathcal{O}(1)$,*
iv) $(\mathbb{P}(E), \mathcal{O}(1))$ *where E is a rk 3 vector bundle over a smooth curve,*
v) $(\mathbb{P}(E), \mathcal{O}(1))$ *where E is a rk 2 vector bundle over a surface of negative Kodaira dimension.*

The above Theorem allows to extend the result of [CF] to 3-folds T which contain a smooth surface H of negative Kodaira dimension such that H is nef and big.

Exercise 5.19.4. *Prove the following. Let T be a terminal 3-fold and $H \subset T$ a smooth surface of negative Kodaira dimension. Assume that H is nef and big, then T is birational to one of the following:*

- $\mathbb{P}^3$
- $H \times \mathbb{P}^1$,
- *a terminal sextic in either* $\mathbb{P}(1,1,1,2,3)$ *or* $\mathbb{P}(1,1,2,2,3)$,
- *a terminal quartic in* $\mathbb{P}(1,1,1,1,2)$,
- *a terminal cubic in* $\mathbb{P}^4$.

There exists a natural geometric interpretation of the conditions imposed in Theorem 5.19.3.

Theorem 5.19.5 ([Me4])**.** *Let $T_d \subset \mathbb{P}^n$ be a degree d non degenerate 3-fold. Suppose that $d < 2n-4$, then any #-Minimal Model $(T^{\#}, \mathcal{H}^{\#})$ of $(T_d, \mathcal{O}(1))$ is in the list of Theorem 5.19.3.*

Proof. Let $\nu : X \to T$ a resolution of singularities and $\mathcal{H} = \nu^*\mathcal{O}(1)$. First prove that $(K_X + H) \cdot H^2 < 0$. We will argue comparing Castelnuovo bound on the genus of C and the genus formula on the surface H. From the latter we obtain that $g(C) = 1 + d/2 + (K_X + H) \cdot C/2$. For the former let $m = \lfloor \frac{d-1}{n-3} \rfloor$ then by Castelnuovo inequality, [GH], we have that

$$g(C) \leq \frac{m(m-1)}{2}(n-3) + m(d-1-m(n-3)).$$

It is therefore enough to impose that

$$1 + d/2 > \frac{m(m-1)}{2}(n-3) + m(d-1-m(n-3)), \tag{5.19.1}$$

after a small calculation one verifies that this is true whenever $d < 2n-4$. Then by adjunction formula, $H \in \mathcal{H}$ is a smooth surface of negative Kodaira dimension and Theorem 5.19.3 applies. □

Remark 5.19.6. This Theorem can be interpreted as the three dimensional counterpart of the classical result that a non degenerate surface $S \subset \mathbb{P}^n$ of degree $d \leq n-1$ is birational either to a rational scroll or to a projective plane, [GH]. Observe that all the 3-fold listed admit an embedding satisfying our numerical criterion.

By means of adjunction theory on terminal varieties, [Me1], one prove an higher dimensional analog of Theorem 5.19.5.

Theorem 5.19.7 ([Me4])**.** *Let $X_d \subset \mathbb{P}^n$ a non degenerate k-fold with $k > 3$ and only $\mathbb{Q}$-factorial terminal singularities. Assume that $d < 2(n-k) - 2$ then a #-minimal model $(X^{\#}, H^{\#})$ of $(X, \mathcal{O}(1))$, in adjunction theory language $(X^{\#}, H^{\#})$ is the first reduction, is one of the following:*

i) *a $\mathbb{Q}$-Fano n-fold of Fano index $1/\rho > k-2$, with $K_{T^\#} \sim -1/\rho H^\#$ and $\Phi_{|H^\#|}$ birational, the complete classification is given in* [Fu2] *if $X^\#$ is Gorenstein and in* [CF] *and* [Sa] *in the non-Gorenstein case.*

ii) *a projective bundle over a curve with fibers $(F, H^\#_{|F}) \simeq (\mathbb{P}^{k-1}, \mathcal{O}(1))$, or a quadric bundle with at most cA_1 singularities, with $H^\#_{|F} \sim \mathcal{O}(1)$;*

iii) *$(\mathbb{P}(E), \mathcal{O}(1))$ where E is a $rk(k-1)$ ample vector bundle either on $\mathbb{P}^2$ or on a ruled surface.*

Remark 5.19.8. Note that since X_d has terminal singularities then, assuming the minimal model conjecture, the above is the classification of #-models of those varieties.

5.19.2. **General elephants of $\mathbb{Q}$-Fano threefolds.** Another direction is the study of the birational class of $\mathbb{Q}$-Fano whose generic section has worse than canonical singularities.

Conjecture 5.19.9 (Reid). Let X be a $\mathbb{Q}$-Fano threefold and $H \in |-K_X|$ a generic section of the anticanonical divisor. Then H has at worst canonical singularities.

The motivation of this conjecture is that to classify Fano variety, as we learned, one uses sections of the fundamental divisor. For non Gorenstein 3-fold with index< 1, this invariant is quite meaningless and one tries to use directly sections of $|-K_X|$. So more than a conjecture is an hope that things are not too bad in this corner of the world. It is to be said that the most recent techniques to study $\mathbb{Q}$-Fano do not rely completely on this procedure. The #-program allows to understand the birational nature of these strange objects.

Theorem 5.19.10 ([Me4]). *Let T be a $\mathbb{Q}$-Fano 3-fold.*
Assume that $dim\phi_{|-K_T|}(T) = 3$ and the general element in $|-K_T|$ has worse than Du Val singularities. Then T is birational to a smooth Fano 3-fold $T^\#$ of index ≥ 2.

The rough idea of the proof is to take a log minimal resolution of $(T, |-K_T|)$ and control the output. By the singularity requirement the generic element in $|-K_T|$ is a uniruled surface therefore we can apply all results of previous sections. For more details confront [Me4].

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